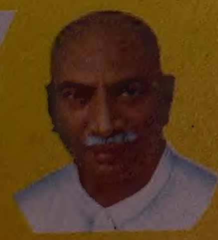
MADURAI KAMARAJ UNIVERSITY

(University with Potential for Excellence)

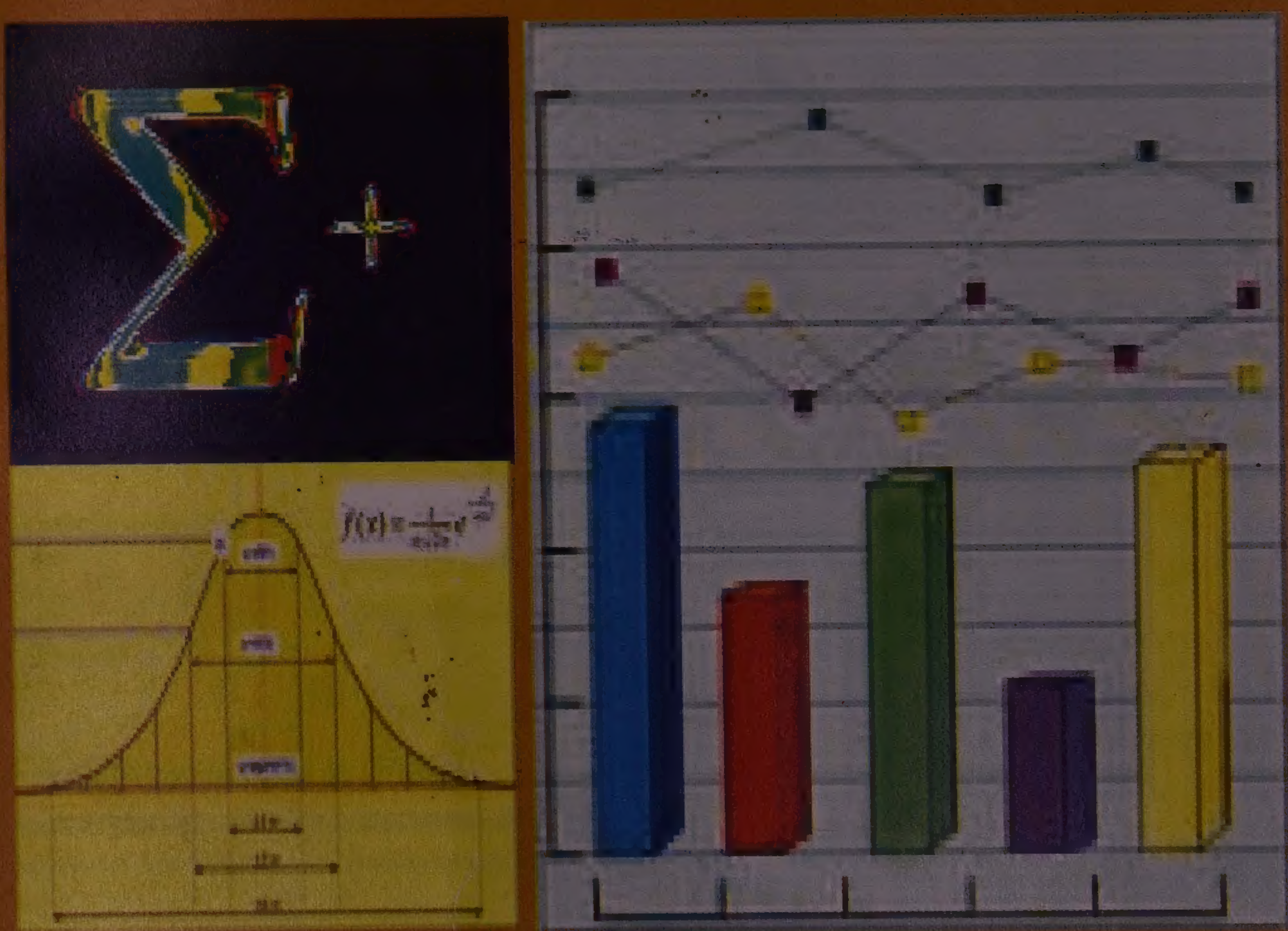


B.A. (Economics)

Second Year

QUANTITATIVE TECHNIQUES

VOLUME - II



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Paper - I

QUANTITATIVE TECHNIQUES

Volume II

Units 4-10

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An Introduction to

'Quantitative Techniques' Volume II

The subject 'Quantitative Techniques' consists of 10 broad divisions and we have already given an outline of what is dealt with in each of these divisions in the 'Introduction to the subject' presented in Volume - I. Out of the 10 divisions, 3 divisions have been given in Volume - I as Unit - 1, Unit - 2 and Unit - 3. The remaining 7 divisions are presented in this Volume - II as Units 4 to 10.

You are advised to read the study materials regularly so that you get 100 in this subject.

Faculty Members.
Dept. of Economics,
DDE, MKU.

CONTENTS

Volume I

Volume II

UNIT – 4

CORRELATION

Space for hints

Introduction

Till now we have been considering such series where various items assumed values of a single variable. There can, however, be such series also where each item assumes the values of two or more variables. For example, suppose the heights and weights of a group of persons are measured. Now we get a series where each item assumes two values-one relating to height and the other relating to weight. Here, each item assumes values of two variables viz., height and weight. If besides heights and weights, the chest measurements were also taken, we get such a series where each item assumes three values-one relating to the variable height the second one relating to the variable weight and the third one relating to the variable chest measurement. Thus, there can be such series where each item assumes the values of a number of variables.

When each item assumes values of more than one variable, there may exist relationship between these variables. A statistical tool useful in finding out whether there is any relationship between the given variables is 'correlation'. The meaning, types and measurement of correlation are described in this Unit - 4.

Unit Objectives :

After studying this Unit, you would be able to understand

* the meaning and types of correlation
* various methods of measurement of correlation
* uses of correlation analysis, particularly in Economics.

Unit Structure :

1. Correlation - Meaning
2. Types of Relationship
3. Correlation represents only general Relationship
4. Spurious Correlation
5. Types of correlation based on nature of Relationship
6. Types based on mathematical forms of Relationship
7 Types based on Number of variables under consideration
8. Degree of Correlation
9. Types of Perfect Correlation

Space for hints

1. Correlation - Meaning

Sometimes it appears that the values of various variables are inter-related. For example, it is likely that the values of the variable 'height of persons' are related to the variable 'weights of the sample' - that is, the weights increase with increase in heights. Such relationships can be found in many types of series, for example, price and supply, sales of umbrellas and the number of rainy days, prices of sugar and sugarcane, ages of husbands and wives, etc.

The **relationship** between two or more variables is called **'correlation'** and the variables are said to be correlated. The relationship between two variables is also called 'covariation'.

2. Types of Relationship

The term 'relationship' can be used in three different senses which we have explained below :

(i) Mutual Dependence

The term 'relationship' can be used in the sense of mutual dependence. Consider the two variables, price and demand. When the price of a commodity decreases, the demand for it increases. On the other hand, when the demand for a commodity increases, its price also increases. Thus, with a change in the values of one of the two variables the values of the other variable also change. Such a relationship between two variables is known as 'mutual dependence' and the variables are said to be 'mutually dependent'.

(ii) Cause and Effect Relationship

The relationship between two variables need not always be the result of their mutual inter-dependence. Changes in the values of one variable may be the cause of change in the values of the other variable; thus there may be cause and effect relationship between the two variables.

Check your Progress

1. What is correlation?

For example, consider the variables 'amount of rainfall' and 'production of foodgrains'. When there is an increase in the amount of rainfall in a given region, then there is an increase in the production of foodgrains in that region. Here the increase in the amount of rainfall is the 'cause' and the increase in the production of foodgrains is the 'effect'. Such a relationship between two variables is known as cause and effect relationship.

Space for hints

Types of cause and effect Relationship :

The cause and effect relationship between two variables may be either direct or indirect. In the example we have considered above, consider one more variable, the price of foodgrains in addition to the two variables viz., the amount of rainfall and the production of foodgrains. When there is an increase in the production of foodgrains the price of foodgrains falls down. Here the increase in the amount of rainfall is the common cause for the increase in the production of foodgrains and the decrease in their price. The cause and effect relationship between rainfall and price of foodgrains is through the third factor 'increase in the production of foodgrains' and thus the relationship is indirect.

(iii) Correlation due to a Common Factor :

Sometimes two variables may not have any type of direct relationship between themselves. Yet we can find that when the value of one variable increases the value of the other variable also increases. This may be due to a third factor which causes the variables to increase in their values.

Consider the two variables-production of paddy and production of cotton. It is obvious that these two variables have no direct relationship among themselves. Yet when there is an increase in the production of paddy, there may be an increase in the production of cotton also. The reason for this may be that there is an increase in the amount of rainfall at various regions. Thus the relationship between the two variables viz., production of paddy and production of cotton, is due to the third factor, viz., the amount of rainfall. This is the third sense in which the word 'relationship' is used.

3. Correlation represents only general relationship

Existence of correlation between two variables is no guarantee that the relationship between the two variables would always be of the same type in each and every occasion. There may be correlation between price and demand so that in **general** whenever there is an increase in price the demand falls, and vice versa. But this dose not mean that whenever price rises demand must fall. It is possible that with a rise in price demand may also go up.

Check your Progress

2. Does correlation mean mutual dependence?

3. Does correlation mean cause and effect relationship?

In the same way, there may be correlation between the heights of husbands and the heights of wives so that in general tall husbands may have

Space for hints

tall wives and short husbands may have short statured wives. This does not mean, every tall husband would have a tall wife. Some tall husbands may have short wives and some short statured husbands may have tall wives.

This is so due to the fact that in economic and social activities various factors affect the data simultaneously; and it is difficult and in most cases it is impossible to study the effects of these factors separately.

4. Spurious Correlation

It may have just happened that the values of two variables increase together or decrease together. But this may be merely a coincidence. For example, consider the two variables, number of radio sets sold and number of suicides. Number of radio sets sold may increase day by day and the number of suicides may also increase. On the basis of the fact that the values of the two given variables increase together, we cannot say that they are correlated. Listening to a radio set has nothing to do with committing suicides. Here the relationship between number of radio sets and number of suicides is called non-sense or spurious correlation'. Unless we prove that the given two variables are logically related, it is better not to establish correlation.

5. Types of correlation based on nature of Relationship

Correlation between two variables can be either positive or negative.

5.1 Positive Correlation:

Suppose there are two variables. Also suppose that with an increase in the value of one variable, the value of the other variable increases and with a decrease in the value of one variable the value of the other variable also decreases. Here the correlation is said to be **'positive'.** Thus, when the values of two variables move in the same direction, correlation is said to be **positive.**

For example, consider the two variables, price of a commodity and the supply of it. When there is an increase in price there is an increase in the supply of the commodity; and when there is a decrease in price there is a decrease in the supply also. Here the values of the two variables, price and supply move in the same direction and hence, the correlation between these two variables is positive. The two variables are said to be positively correlated.

Check your Progress

4. What do you understand by positive correlation? Give an example.

5. What is negative correlation? Give an example.

5.2 Negative Correlation:

Suppose there are two variables. Also suppose that with an increase in the value of one variable, the value of the other variable decreases; with the decrease in the value of one variable the value of the other variable also increases. In such a case, the correlation is said to be **'negative'**. Thus, when

the values of two variables move in the opposite direction, correlation is said to be **'negative'**.

Space for hints

For example, consider the two variables, price and demand for a commodity, When there is an increase in price of the commodity the demand for it decrease- and when there is a decrease in price the demand for it increases. Here the values of the two variables, price and demand move in the opposite direction. Hence, the correlation between these two variables is negative. The two variables are said to be negatively correlated.

6. Types based on mathematical forms of Relationship

Correlation between two variables may also be of the following two types:

(i) Linear Correlation.

(ii) Non-linear or curvi linear Correlation.

6.1 Linear Correlation:

Suppose whenever there is a 5% increase in the price of a commodity there is a 10% increase in the supply of that commodity. That is, the values of the two variables, price and supply of a commodity change in the constant ratio, 1:2. In such cases, where the values of two variables vary in a constant ratio, correlation is said to be 'linear'.

6.2 Non-linear or curvi-linear Correlation:

If the amount of change in the value of one variable and the corresponding amount of change in the value of the other variable are not in a **constant ratio**, the correlation is said to be **'non-linear or curvi-linear'.** Suppose, for a 5% increase in price there is a 20% increase in supply, and for a second 5% increase in price there is only 15% increase in supply. In the first case, the ratio between the amounts of change in price and supply is 1: 4 (= 5:20) in the second case the ratio is 1 : 3 (=5:15). In this case, the correlation is non-linear because the ratio between the amounts of change in price and supply is not constant but variable.

Here, we confine ourselves to the study of only linear correlation.

7 Types of correlation based on Number of Variables

7.1 Simple Correlation:

If only two variables are considered and if correlation exists between these two variables, the correlation is said to be **'simple'**.

Check your Progress

6. What is linear correlation?

7. What is non-linear correlation?

Space for hints

7.2 Multiple Correlation:

If we consider more than two variables, then the association of one variable with the rest of the variables is called "multiple correlation". Demand for a commodity depends upon the price of it, income of the consumer, taste of the consumer, etc. Here the association of demand with the rest of the variables viz., price, income, taste, etc. is called 'multiple correlation'.

7.3 Partial Correlation:

If we consider more than two variables and if the relationship between only two variables is considered assuming the values of other related variables to remain constant, then the correlation is said to be 'partial'. Demand is related to price, income, taste. Here if we study the relationship between demand and price alone assuming income, taste etc., to remain constant the correlation is said to be 'partial'.

In our lesson, we confine ourselves to the study of simple, linear correlation.

8. Degree of Correlation

There are two extreme cases of correlation between two variables. They are explained below.

8.1 No Correlation - Nil Dependence

There may not be any relationship at all between two variables. For example, there is no relationship at all between the number of cars registered with the total number of births recorded during a period of years. In such cases, we say that there is absence of correlation or no correlation between the two variables.

8.2 Perfect Correlation - Complete Dependence

At the other extreme, one variable may be completely dependent on the other variable. For example, the circumference of a circle completely depends on the length of diameter. When there is complete dependence of one variable on the other, we say that one is the function of the other. When one variable is the function of the other, the values of one variable bear a perfectly definite ratio to the value of the other. Here we say that the correlation between the other two variables is perfect. So, perfect correlation is the other extreme.

Check your Progress

8. Do you know the difference between simple and multiple correlation?

9. Give an example of partial correlation in Economics.

10. What do you understand by perfect correlation?

9. Types of Perfect Correlation

Space for hints

There are two types of perfect correlation. They are explained below.

9.1 Perfect Positive correlation

When an increase in the value of one variable results in an increase in the other variable in a fixed proportion, the correlation is said to be perfect positive. For example, when the diameter of a circle increases, its circumference also increases in a fixed proportion viz., $1:\pi$. Here the correlation is perfect positive.

9.2 Perfect Negative correlation

When an increase in the value of one variable results in a decrease in the value of the other variable in a definite ratio, the correlation is said to be perfect negative. For example, in the case of gases obeying Boyle's law, volume decreases in a definite ratio when the pressure increases.

10. Limited Degree of Correlation - Imperfect Correlation

Variables completely dependent on each other. (i.e. variables which are perfectly correlated) are found only in exact sciences like Mathematics, Physics etc. In the case of inexact sciences like Economics, we cannot find variables which are completely dependent on each other. Hence, in economic data perfect correlation is usually not found. In such cases, correlation is not perfect; that is, correlation exists only to a limited extent.

When there is limited degree of correlation, the change in one variable does not bear a definite ratio to the change in the other. For instance, when advertisement expenditure is increased by Rs.1000, the number of clients may increase by 20. When the advertisement expenditure is increased again by another Rs.1000, it may not result in another 20 new clients; instead, it may be more than 20 (or less than 20). Thus, the changes in the two variables are not in a definite ratio.

11. Types of Limited Degree Correlation

Limited degree correlation is also of two types. They are

(i) Limited degree of positive correlation :

When an increase in one variable results in an increase in the other variable also, but not in a definite ratio, it is called limited degree of positive correlation. For instance, an increase in the price of a commodity will result in an increase in the supply of it. But the two increases need not be in a fixed ratio.

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(ii) Limited degree of negative correlation :

When an increase in the value of one variable results in a decrease in the other variable but not in a definite ratio, it is called limited degree of negative correlation. For instance, an increase in the price of a commodity increases, the demand for it decreases but not in a definite ratio.

12. Methods of mesurement of correlation

Correlation can be studied using any one of the following methods:

(i) Scatter diagram

(ii) Correlation graph

(iii) Karl Pearson's coefficient of correlation; and

(iv) Rank correlation coefficient.

The first method given above is graphical method of studying correlation. With the help of this method we can find out only the nature of correlation. That is, we can find out whether correlation is positive or negative. But we cannot find out exactly the extent of correlation between the two variables.

The last two methods give us exactly the magnitude of correlation in numerical terms.

13. Scatter Diagram

Scatter Diagram is a graphical method and it is the simplest method of studying correlation between two related variables.

13.1 Method of construction of Scatter Diagram

Suppose, each item of the given data assumes values of two variables x and y. We mark the values of the variable x on the x-axis and values of the variable y on the y-axis. We plot a point against each value of y and the corresponding value of x given. We get a **swarm of dots** and this we call '**the scatter diagram**' or '**dot diagram**' The concentration and scatter of dots in the diagram will give us an idea about the correlation between the two variables given.

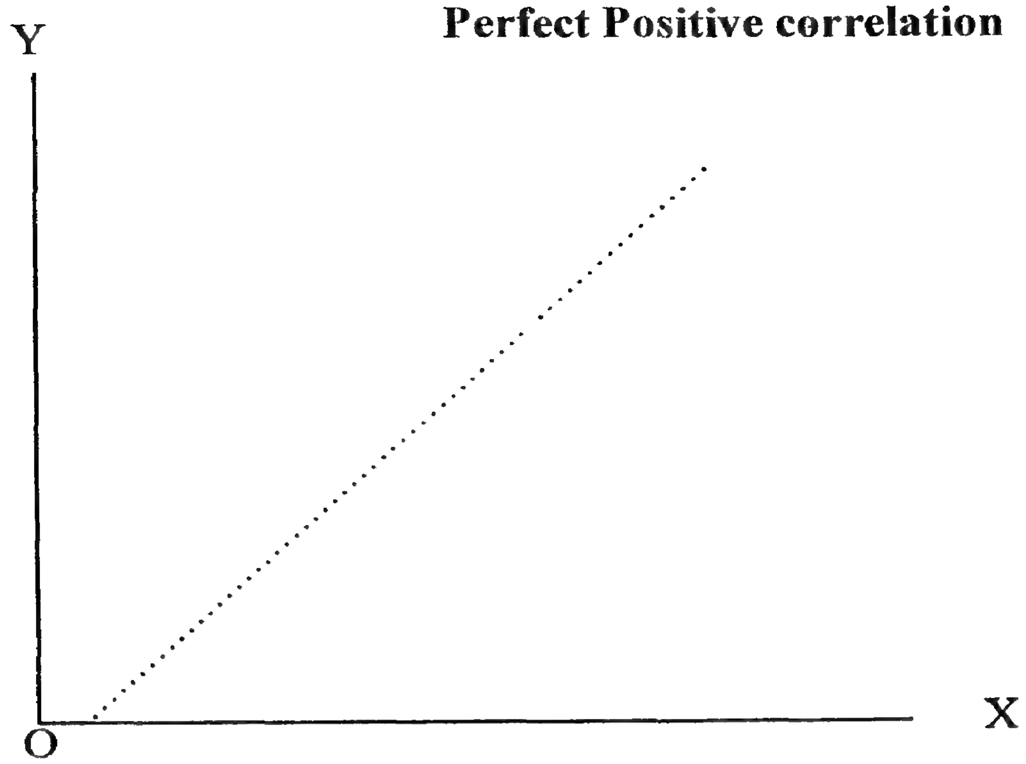
(i) Perfect Positive correlation :

If the plotted points or dots form a **straight line running** from left to right in the **upward direction**, correlation is said to be '**perfect positive**'. In the figure below, we have given the scatter diagram when there is perfect positive correlation.

Check your Progress

11. What is scatter diagram?

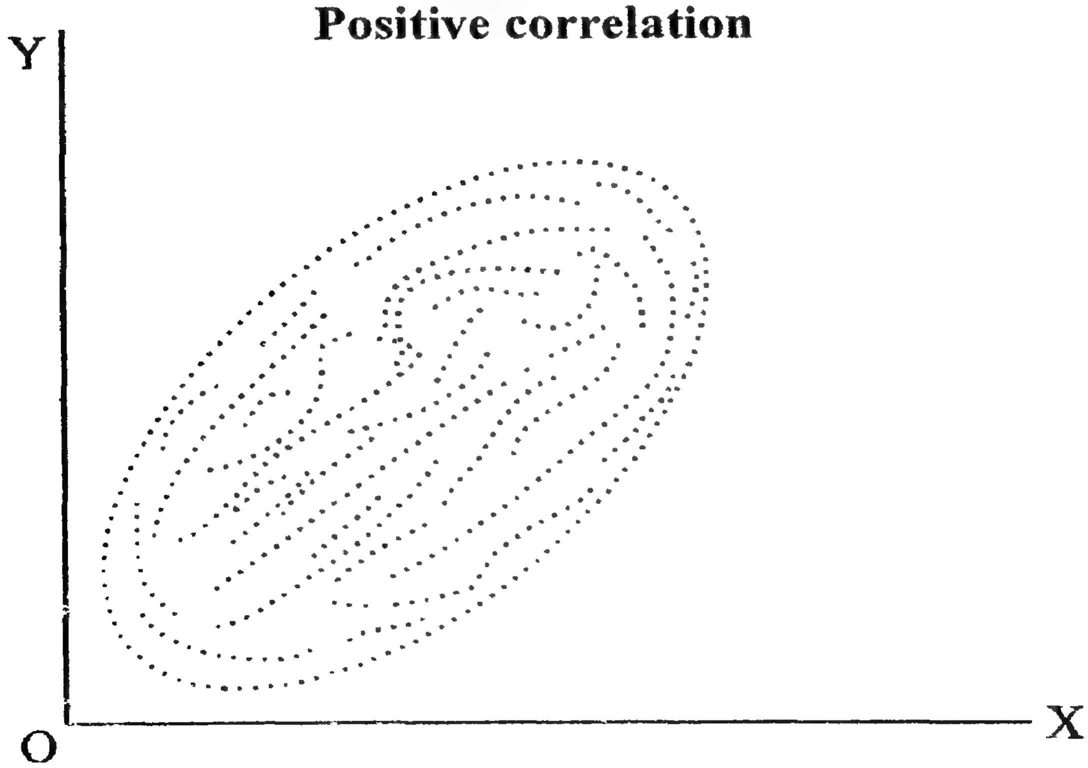
Fig. - 4.1
Perfect Positive correlation



(ii) Positive correlation :

If the dots are scattered around a **straight line running from left to right** in the upward direction, then the correlation between the given variables is said to be **positive**. The figure below shows the scatter of points when there is positive correlation.

Fig. - 4.2
Positive correlation



(iii) Perfect Negative correlation :

Suppose, the dots in the scatter diagram form a **straight line running**

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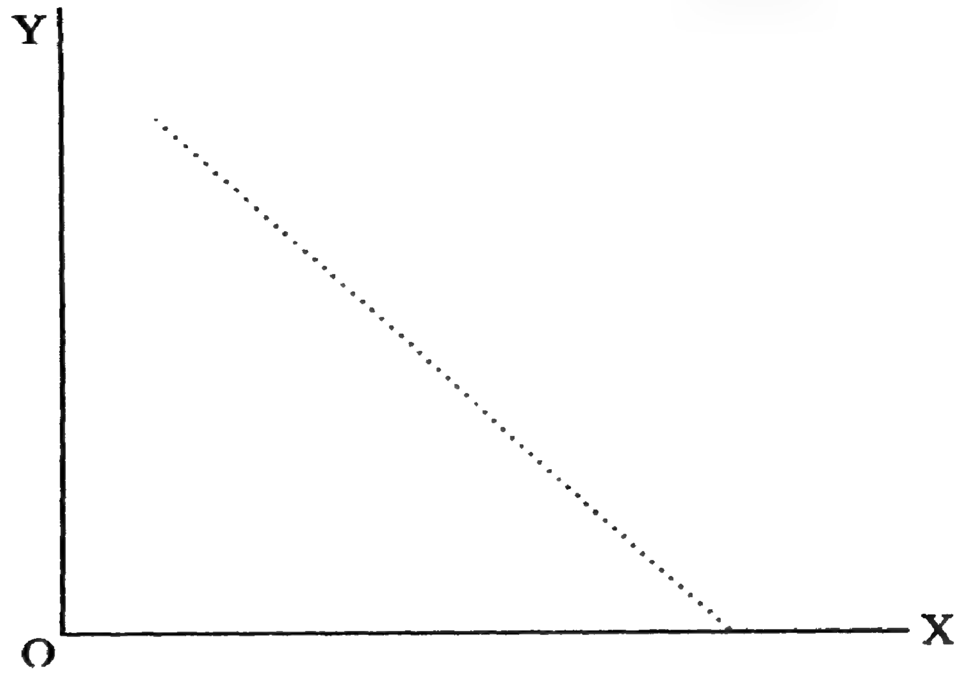
from left to right in the downward direction. In this case, correlation is said to be '**perfect negative**'. We have given in below, the scatter corresponding to 'perfect negative correlation'.

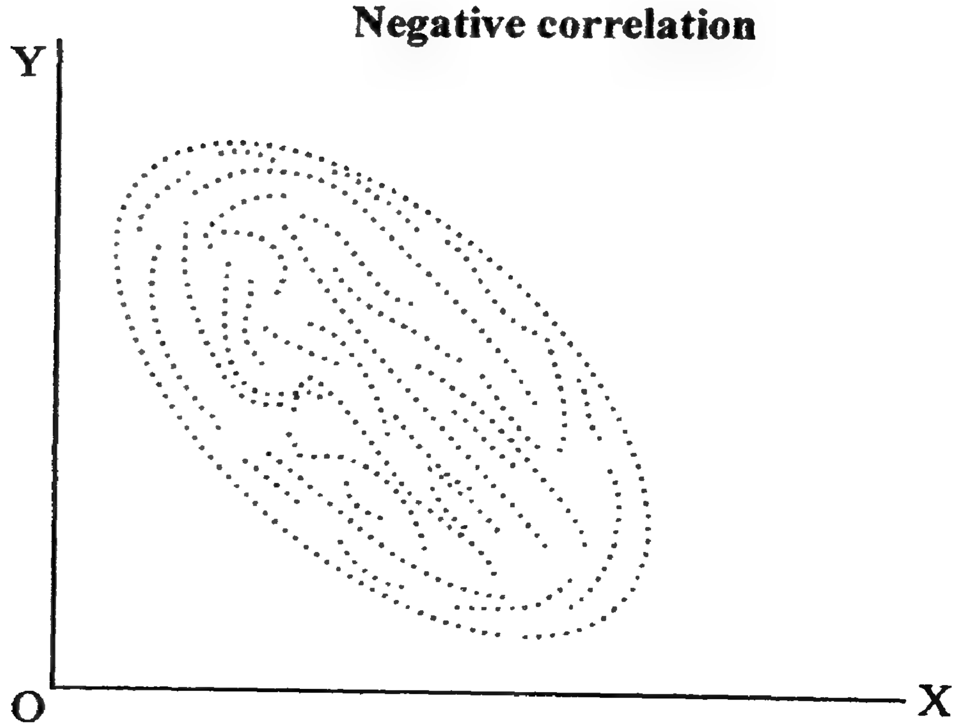
Fig. - 4.3
Perfect Negative correlation



(iv) Negative correlation :

Suppose, in the scatter diagram, the dots are scattered **around a straight line running from left to right** in the downward direction. In this case, correlation is said to be '**negative**'. In the following figure, we have given the form of scatter of points when there is negative correlation.

Fig. - 4.4
Negative correlation



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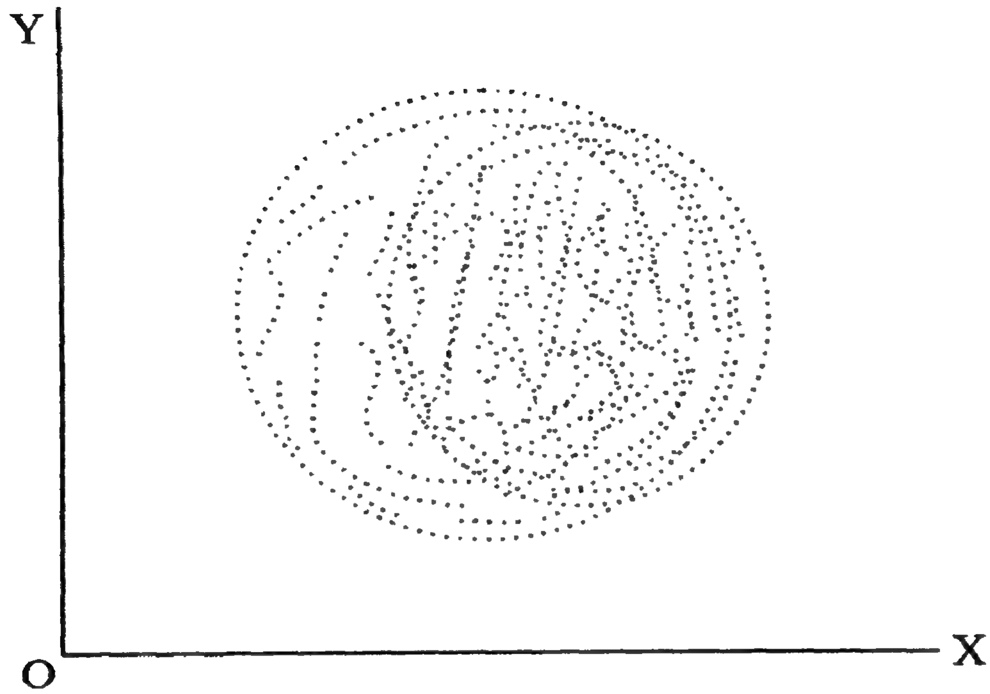
(v) No correlation :

In some cases the plotted points **do not form a straight line and do not even form a flow of points** from left to right in the upward or downward direction. The dots lie scattered all over the diagram as shown in the figure given above. In this case, it is said that there is absence of correlation and the variables given are **not correlated.**

Fig. - 4.5
Absence of correlation



13.2 Merits and Demerits of scatter diagram:

Scatter diagram is not based on any mathematical concept and hence it is easy to understand even by the layman.

It is the simplest and quickest method of studying the presence of correlation between two variables.

Against the merits we have stated above it has the drawback that it helps us only to know the presence and direction of correlation. It does not provide us a measure of the degree of correlation present.

14. Correlation graph:

Correlation graph is used where vari bles are given with reference to a peirod of time. To draw the correlation graph, we mark time on the x-axis and the values of the variables given on the y-axis.

Consider the table given below. In this table we have given the values of two variables viz., income annd expenditure in each year for a period of ten years from 1951 to 1960.

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Year	Income (Rs.)	Expenditure (Rs.)
1951	100	90
1952	105	100
1953	125	110
1954	110	105
1955	115	112
1956	103	98
1957	118	108
1958	106	95
1959	120	115
1960	130	120

To draw the correlation graph of the data given above, we mark the years from 1951 to 1960 on the x-axis. We mark the values of both income and expenditure in rupees on the y-axis.

Above the year 1951 marked on the x-axis and against the income Rs.100 marked on the y-axis, we plot a point.

Above the year 1952 and against the income Rs.105 we mark a point and so on.

All such points plotted are joined by a line which gives the graph of income. In the figure we have given below, the income graph is denoted by A.

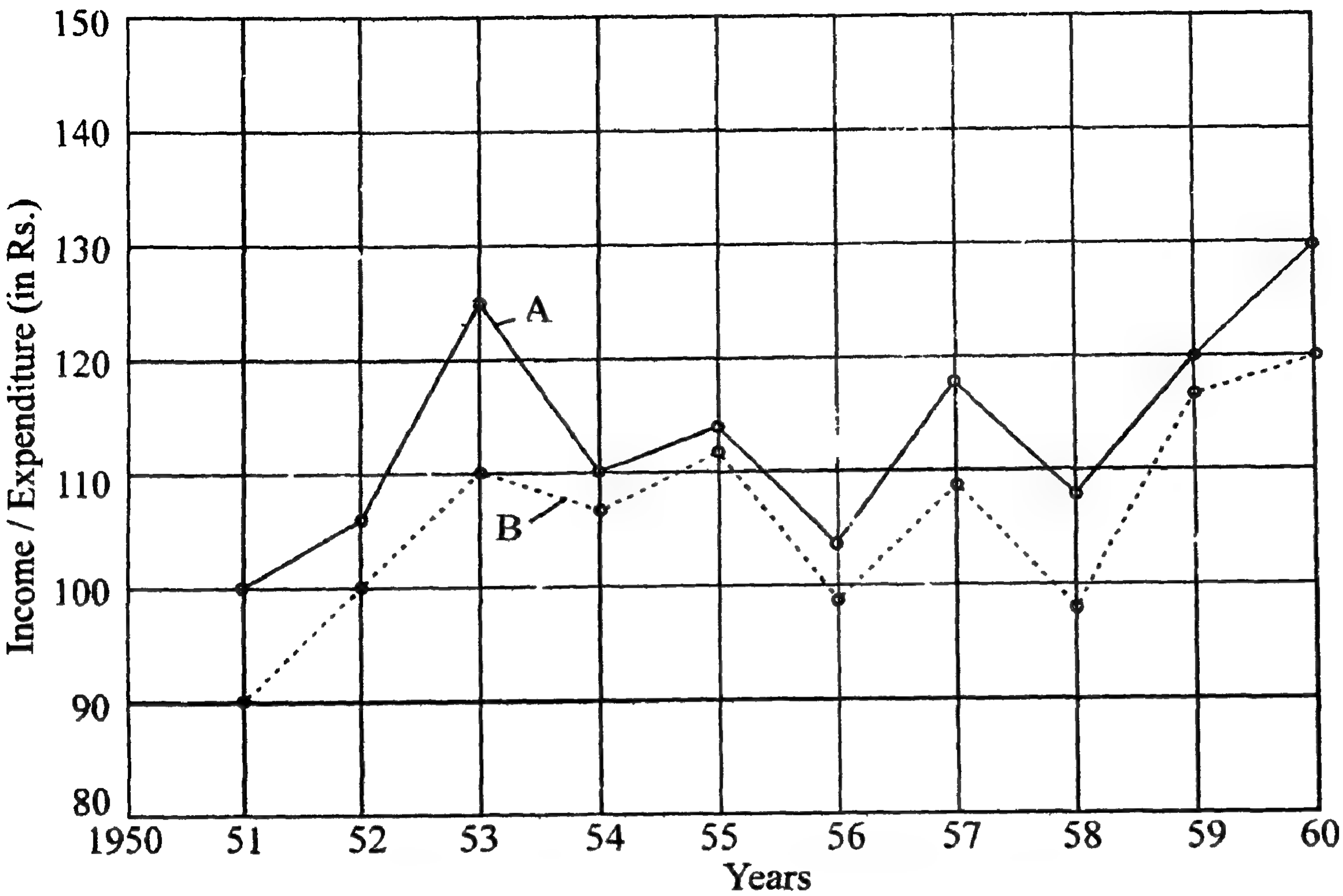
In the same way. the expenditure graph is obtained as follows:

Above the year 1951 marked on the x-axis and against the expenditure Rs.90 marked on the y-axis, we plot a point.

Above the year 1952 and against the expenditure Rs.100. We plot a point and so on.

All such points, plotted are joined by a line which gives the graph of expenditure. In the figure we have given below the expenditure graph is denoted by B. To distinguish expenditure graph from the income graph, we have given the expenditure graph by dotted lines.

Correlation graph representing income and expenditure for the year 1951 to 1960



A - Income graph   B - Expenditure graph

In the correlation graph, the curves of the two variables are very close to each other and if they move in the same direction, the variables are positively related. On the other hand, if the curves of the two variables move in opposite directions, the variables are negatively correlated.

In our example, the curves of income and expenditure move in the same direction and hence they are positively correlated.

Correlation graph method also gives us only an approximate idea of the correlation present between two variables and it lacks numerical measure.

15. Karl Pearson's Coefficient of Correlation

Coefficient of correlation is calculated to study the extent or degree of correlation between two variables. Coefficient of correlation gives us the degree of correlation in quantitative terms, Karl Pearson has given the formula to calculate the coefficient of correlation and therefore it is called Karl Pearson's coefficient of correlation.

Check your Progress

12. Who developed coefficient of correlation? Give its formula.

15.1 Assumptions :

i) The correlation between the two given variables is assumed to be linear.

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ii) The factors affecting the two given variables are assumed to be related to each other in a relationship of cause and effect.

iii) The values of the two given variables are affected by multiple causes which are common to both of them.

15.2 Definition :

When the values of two variables x and y are given Karl Pearson has given the formula to calculate the correlation coefficient as follows:

$$r = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{n\,\sigma_x\,\sigma_y}$$

Where r denotes the coefficient of correlation;

x and y are pairs of values of the variables x and y

$\bar{x}$ is the mean of the values of the variable x

$\bar{y}$ is the mean of the values of the variable y.

σ_x is the standard deviation of the values of the variable x (i.e) $\sigma_x = \sqrt{\frac{\Sigma(x-\bar{x})^2}{n}}$

σ_y is the standard deviation of the values of the variable y (i.e) $\sigma_y = \sqrt{\frac{\Sigma(y-\bar{y})^2}{n}}$

n is the total number of pairs of values of the variables x and y given.

15.3 The detailed procedure to calculate the coefficient of correlation :

Consider the following series of values of two variables x and y.

x	y
5	8
7	9
3	5
1	4
9	9
12	13
8	7
3	9

Step -1 :

We find out the mean of the values of x separately, and the mean of the values of y separately.

We denote the mean of x by $\bar{x}$ and the mean of y by $\bar{y}$

$$\therefore \bar{x} = \frac{\text{Sum of the values of x}}{\text{Number of values of x}}$$

$$= \frac{\Sigma x}{n}$$

$$= \frac{5+7+3+1+9+12+8+3}{8}$$

$$= \frac{48}{8} = 6$$

$$\bar{y} = \frac{\text{Sum of the values of y}}{\text{Number of values of y}}$$

$$= \frac{\Sigma y}{n}$$

$$= \frac{8+9+5+4+9+13+7+9}{8}$$

$$= \frac{64}{8} = 8$$

Step -2 :

Find out the deviation of each value of x from its mean. $\bar{x}$ and deviation of each value of y from its mean, $\bar{y}$. We denote the deviations of the values of x by $(x-\bar{x})$ and the deviations of the values of y by $(y-\bar{y})$.

In our example,

The deviation of the first value of x from $\bar{x}$ $= 5-6 = -1$

The deviation of the second value of x from $\bar{x}$ $= 7-6 = 1$

The deviation of the third value of x from $\bar{x}$ $= 3-6 = -3$

and so on.

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Space for hints

These deviations are the values of $(x-\bar{x})$

∴ The values of $(x-\bar{x})$ are

–1 (= 5–6), 1(= 7–6), –3(= 3 –6), –5(= 1-6).

3(= 9–6), 6(= 12–6), 2(= 8 –6) and –3(= 3-6).

Now, the deviation of the first value of y from $\bar{y}$ = 8–8 = 0

The deviation of the second value of y from $\bar{y}$ = 9–8 = 1

and so on.

These deviations are the values of $(y-\bar{y})$

∴ The values of $(y-\bar{y})$ are

0 (= 8–8), 1(= 9–8), –3(= 5 –8), –4(= 4-8).

1 (= 9–8), 5(= 13–8), -1(= 7 –8) and 1(= 9-8).

Step -3 :

Find out the values of the product $(x-\bar{x})\ (y-\bar{y})$. To get the values of $(x-\bar{x})(y-\bar{y})$, the first value of $(x-\bar{x})$ is multiplied by the first value of $(y-\bar{y})$, the second value of $(x-\bar{x})$ is multiplied by the second value of $(y-\bar{y})$ and so on.

In our example,

the first value of $(x-\bar{x}) = -1$

the first value of $(y-\bar{y}) = 0$

their product $=(-1) \times 0 = 0$

the second value of $(x-\bar{x}) = 1$

the second value of $(y-\bar{y}) = 1$

their product $= 1 \times 1 = 1$

and so on.

∴ The values of $(x-\bar{x})(y-\bar{y})$ are

[(–1)×0], [1×1], [(–3)×(–3)], [(–5)×(–4)], [3×1], [6×5], [2×(–1)], [(–3) ×1]

That is, $(x-\bar{x})(y-\bar{y})$ values are : 0, 1, 9, 20, 3, 30, –2, –3.

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Step -4 :

Sum-up the values of $(x-\overline{x})(y-\overline{y})$

We denote this sum by $\Sigma(x-\overline{x})(y-\overline{y})$

In our example,

$$\begin{aligned}\Sigma(x-\overline{x})(y-\overline{y}) &= 0+1+9+20+3+30-2-3\\ &= 63-5\\ &= 58\end{aligned}$$

Step -5 :

Find out the standard deviations of the values of x and of the values of y separately. We denote the standard deviation of x by σx and the standard deviation of y by σy.

To find out the value of σx, first we find out the squares of the values of $(x-\overline{x})$ and sum up. That is, we find out the values of $(x-\overline{x})^2$ and sum up. We denote the sum of the values of $(x-\overline{x})^2$ by $\Sigma(x-\overline{x})^2$

Then using the formula,

$\sigma_x = \sqrt{\dfrac{\Sigma(x-\overline{x})^2}{n}}$ the value of σx is calculated. Here n denotes the total number of pairs of values of x and y given. In our example,

$n = 8$

The values of $(x-\overline{x})^2$ are

$1\ [= (-1)^2\ ],\ 1[= (1)^2],\ 9[= (-3)^2],\ 25\ [= (-5)^2],$

$9\ [= (3)^2\ ],\ 36[= (6)^2],\ 4[= (-2)^2]$, and $9\ [= (-3)^2]$.

$\therefore\ \Sigma(x-\overline{x})^2 = 1+1+9+25+9+36+4+9 = 94$

$$\therefore\ \sigma_x = \sqrt{\frac{\Sigma(x-\overline{x})^2}{n}} = \sqrt{\frac{94}{8}}$$

Space for hints

In the same way, the value of , is got by finding out values of (

and their sum, denoted by $\Sigma(y-\bar{y})^2$ and then using the formula,

$$\sqrt{\frac{\Sigma(y-\bar{y})^2}{n}}$$

... ple, n = 8

... values of $(y-\bar{y})^2$ are

0 [= $(0)^2$], 1[= $(1)^2$], 9[= $(-3)^2$], 16 [= $(-4)^2$], 1 [= $(1)^2$], 25[= $(5)^2$], 1[= $(-1)^2$],1[= $(1)^2$].

$\therefore\ \Sigma(y-\bar{y})^2 = 0+1+9+16+1+25+1+1 = 54$

$$\sigma_y = \sqrt{\frac{\Sigma(y-\bar{y})^2}{n}} = \sqrt{\frac{\ }{8}}$$

Step -6 :

Knowing the values of $\Sigma(x-\bar{x})(y-\bar{y})$, n, σx and σy we get the value of r using the formula,

$$r = \frac{\Sigma(x-\bar{x})(x-\bar{y})}{n\ \sigma x\ \sigma y}$$

In our example,

$\Sigma($

$$\sigma_x = \sqrt{\frac{94}{8}}$$

$$\sigma_y = \sqrt{\frac{54}{8}}$$

$$\therefore r = \frac{58}{8\times\sqrt{\frac{94}{8}}\times\sqrt{\frac{54}{8}}}$$

$$r = \frac{58}{8\times\sqrt{\frac{94}{8}\times\frac{54}{8}}}$$

Space for hints

$$r = \frac{58}{8 \times \frac{\sqrt{94 \times 54}}{8^2}}$$

$$= \frac{58}{\sqrt{94 \times 54}}$$

Value of $\sqrt{94 \times 54}$ is got with the help of log table as follows:

$$\log 94 = 1.9731$$

$$\log 54 = 1.7324$$

$$\log 94 + \log 54 = 1.9731 + 1.7324$$

$$\frac{\log 94 + \log 54}{2} = \frac{3.7055}{2} = 1.85275$$

$$\text{Anti-log } (1.85275) = 71.24$$

$$\sqrt{94 \times 54} = 71.24$$

$$r = \frac{58}{71.24}$$

$$= 0.814$$

$$\text{Coefficient of correlation} = 0.814$$

We can give the various steps involved in the calculation of correlation (of two variables x and y) in a summarised form as follows:

(i) Mean of the values of the variable x and the mean of the values of the variable y are found out separately. These means are denoted by $\bar{x}$ and $\bar{y}$ respectively.

(ii) Deviation of each value of x from $\bar{x}$ is found out and is denoted by $(x-\bar{x})$.

Deviation of each value of y from $\bar{y}$ is found out and is denoted by $(y-\bar{y})$.

(iii) Each value of $(x-\bar{x})$ is multiplied by the corresponding value of $(y-\bar{y})$ and the product is denoted by $(x-\bar{x})(y-\bar{y})$.

(iv) We denote the total number of pairs of values of x and y given by n.

Space for hints

(v) Standard deviation of the values of x, denoted by σ_x is calculated using the formula,

$$\sigma_x = \sqrt{\frac{\Sigma(x-\bar{x})^2}{n}}$$

Standard deviation of the values of y, denoted by σ y is calculated using the formula,

$$\sigma_y = \sqrt{\frac{\Sigma(y-\bar{y})^2}{n}}$$

(vi) Now knowing the values of $\Sigma(x-\bar{x})(y-\bar{y})$, n, σx and σy we calculate the value of correlation coefficient using the formula,

$$r = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{n\sigma x\sigma y}$$

where r denotes the correlation coefficient.

15.4 Alternative formula to calculate the correlation coefficient :

$$r = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\sqrt{\left[\Sigma(x-\bar{x})^2 \times (y-\bar{y})^2\right]}}$$

If we use the above formula, we need not find out the values of σx and σy separately. So, for the calculation of the value of r we use the above formula rather than the formula we have given earlier. When we use this formula also, the first four steps remain the same. Only in the fifth step there is a change.

The fifth step is as follows:

We calculate the squares of the values of $(x-\bar{x})$ [i.e] we calculate the values $(x-\bar{x})^{of\ 2}$. These values of $(x-\bar{x})^2$ summed up and the sum is denoted by $\Sigma(x-\bar{x})^2$.

We calculate the squares of the values of $(y-\bar{y})$ [i.e] we calculate the values of $(y-\bar{y})^2$. These values of $(y-\bar{y})^2$ summed up and the sum is denoted by $\Sigma(y-\bar{y})^2$.

Now knowing the values of $\Sigma(x-\bar{x})(y-\bar{y})$, $\Sigma(x-\bar{x})^2$ and $\Sigma(y-\bar{y})^2$, we calculate the value of correlation coefficient using the formula,

$$r = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\sqrt{\left[\Sigma(x-\bar{x})^2 \times (y-\bar{y})^2\right]}}$$

Example - 1

Space for hints

Calculate the correlation coefficient from the following data:

x	y
10	9
6	4
9	6
10	9
12	11
13	13
11	8
9	4

All the necessary calculations for the computation of r may be done in a table as shown below

$\bar{x} = 10$ $\bar{y} = 8$

x	$(x-\bar{x})$	$(x-\bar{x})^2$	y	$(y-\bar{y})$	$(y-\bar{y})^2$	$(x-\bar{x})(y-\bar{y})$
10	0	0	9	1	1	0
6	–4	16	4	–4	16	16
9	–1	1	6	–2	4	2
10	0	0	9	1	1	0
12	2	4	11	3	9	6
13	3	9	13	5	25	15
11	1	1	8	0	0	0
9	–1	1	4	–4	16	4
80		32	64		72	43

n = 8 $\Sigma x = 80$, $\Sigma y = 64$

$$\therefore \bar{x} = \frac{\Sigma x}{n} = \frac{80}{8} = 10$$

$$\therefore \bar{y} = \frac{\Sigma y}{n} = \frac{64}{8} = 8$$

Space for hints

$$\Sigma(x-\bar{x})^2 = 32$$

$$\Sigma(y-\bar{y})^2 = 72$$

$$\Sigma(x-\bar{x})(y-\bar{y}) = 43$$

$$\therefore r = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\sqrt{\left[\Sigma(x-\bar{x})^2 \times (y-\bar{y})^2\right]}}$$

$$= \frac{43}{\sqrt{32 \times 72}}$$

$$= \frac{43}{\sqrt{2304}}$$

$$= \frac{43}{\sqrt{48 \times 48}}$$

$$= \frac{43}{48} = .9 \text{ (approx.)}$$

Answer :

Correlation coefficient = .9

Example - 2

x	y
13	39
14	40
15	43
16	34
17	36
18	39
19	48
20	47
21	52

Necessary calculations for the computation of r are shown in the following table:

$\bar{x} = 17$ $\bar{y} = 42$

$\bar{x}$	$(x-\bar{x})$	$(x-\bar{x})^2$	y	$(y-\bar{y})$	$(y-\bar{y})^2$	$(x-\bar{x})(y-\bar{y})$
13	–4	16	39	–3	9	12
14	–3	9	40	–2	4	6
15	–2	4	43	1	1	–2
16	–1	1	34	–8	64	8
17	0	0	36	–6	36	0
18	1	1	39	–3	9	–3
19	2	4	48	6	36	12
20	3	9	47	5	25	15
21	4	16	52	10	100	40
153		60	378		284	88

$$\therefore \bar{x} = \frac{\Sigma x}{n} = \frac{153}{9} = 17$$

$$\therefore \bar{y} = \frac{\Sigma y}{n} = \frac{378}{9} = 42$$

$$\Sigma (x-\bar{x})^2 = 60$$

$$\Sigma (y-\bar{y})^2 = 284$$

$$\Sigma (x-\bar{x})(y-\bar{y}) = 88$$

$$\therefore r = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\sqrt{\Sigma(x-\bar{x})^2 \times \Sigma(y-\bar{y})^2}}$$

$$= \frac{88}{\sqrt{60 \times 284}}$$

$$= \frac{88}{\sqrt{17040}}$$

value of $\sqrt{17040}$ is found out using log tables as follows:

$$\log 17040 = 4.2314$$

Space for hints

$$\log \sqrt{17040} = \frac{\log \sqrt{17040}}{2}$$

$$= \frac{4.2314}{2}$$

$$= 2.1157$$

$$\text{Anti-log } (2.1157) = 130.5$$

$$\therefore \sqrt{17040} = 130.5$$

$$\therefore r = \frac{88}{130.5}$$

$$= 0.674$$

15.5 Short-cut Method* :

The procedure to calculate the correlation coefficient by the short-cut method can be given as follows:

(i) Some value within the range of the values of the given variable x is chosen as the assumed average for x. We denote this assumed average by A.

Deviation of each value of x from A is found out. That is, the values of (x–A) are found out. We denote (x–A) by d_x

(ii) Similarly, Some value within the range of the values of the given variable y is chosen as the assumed average for y. We denote this value by B.

Deviation of each value of y from B is found out. That is, the values of (y–B) are found out. We denote (y–B) by d_y

(iii) All the values of d_x are summed up and the sum is denoted by Σd_x

All the values of d_y are summed up and the sum is denoted by Σd_y

(iv) Each value of d_x is multiplied by the corresponding value of d_y . That is, values of the product $d_x d_y$ are found out.

These values are summed up. We denote this sum by $\Sigma d_x d_y$

*Note: In the examination, if it is asked to calculate r, you can use either the Direct method or the Short-cut method. You need not calculate r by both the methods. But the answer must be the same regardless of the method adopted by you.

Space for hints

(v) Square of each value of d_x is found out. That is, the values of d_x^2 are found out. These values of d_x^2 are summed up and the sum is denoted by Σd_x^2

Square of each value of d_y is found out. That is, the values of d_y^2 are found out. These values of d_y^2 are summed up and the sum is denoted by Σd_y^2.

(vi) Total number of pairs of values of x and y given is found out. We denote this number by n.

(vii) Now the correlation coefficient is calculated using the following formula

$$r = \frac{\left(n \times \Sigma d_x d_y - (\Sigma d_x)(\Sigma d_y)\right)}{\sqrt{\left[n \times \Sigma d_x^2 - (\Sigma d_x)^2\right] \times \left[n \times \Sigma d_y^2 - (\Sigma d_y)^2\right]}}$$

Example -3:

Calculate the coefficient of correlation from the pairs of values given below by the short-cut method.

x	y
28	23
41	34
40	33
38	34
35	30
33	26
40	28
32	31
36	36
33	38

Maximum value of x = 41

Minimum value of x = 28

Thus the values of x range from 28 to 41.

Therefore, some value in between 28 and 41 is taken as the assumed average of x viz., A.

Space for hints

Let us take 35 as the value of A ∴ A = 35

Similarly, Maximum value of y = 38 Minimum value of y = 23

∴ We take some value in between 23 and 41 is taken as the assumed average of y.

Let us take 30 as B.

∴ B = 30

All the necassary calculations are shown in the following table

Let A = 35 and B = 30

x	$(x-A)=d_x$	d_x^2	y	$(y-B)=d_y$	d_y^2	d_xd_y
28	–7	49	23	–7	49	49
41	6	36	34	4	16	24
40	5	25	33	3	9	15
38	3	9	34	4	16	12
35	0	0	30	0	0	0
33	–2	4	26	–4	16	8
40	5	25	28	–2	4	–10
32	–3	9	31	1	1	–3
36	1	1	36	6	36	6
33	–2	4	38	8	64	–16
Total	6	162		13	211	85

$\Sigma d_x = 6$; $\Sigma d_y = 13$; $\Sigma d_xd_y = 85$; $\Sigma d_x^2 = 162$; $\Sigma d_y^2 = 211$

n = Total number of pairs of values of x and y given

= 10

$$r = \frac{\left(n\times\Sigma d_xd_y - (\Sigma d_x)(\Sigma d_y)\right)}{\sqrt{\left[n\times\Sigma d_x^2 - (\Sigma d_x)^2\right]\times\left[n\times\Sigma d_y^2 - (\Sigma d_y)^2\right]}}$$

$$= \frac{10\times 85 - (6\times 13)}{\sqrt{\left[10\times 162 - (6)^2\right]\times\left[10\times 211 - (13)^2\right]}}$$

$$= \frac{850-78}{\sqrt{[1620-36]\times[2110-169]}}$$

$$= \frac{772}{\sqrt{1584\times1941}}$$

We find out the value of $\sqrt{1584\times1941}$ using the log table as follows:

$$\log 1584 = 3.1998$$

$$\log 1941 = 3.2880$$

$$\log 1584 + \log 1941 = 3.1998+3.2880$$

$$= 6.4878$$

$$\frac{\log 1584 + \log 1941}{2} = \frac{6.4878}{2}$$

$$= 3.2439$$

$$\text{Anti-log } (3.2439) = 1754$$

$$\therefore \sqrt{1584\times1941} = 1754$$

$$\therefore r = \frac{772}{\sqrt{1584\times1941}}$$

$$= \frac{772}{1754}$$

$$= 0.4401$$

Answer :

Correlation coefficient = .4401

Example 4:

Find how heights and weights are correlated from the following data.

Height (in inches)	Weight (in inches)
57	113
59	117
62	126
63	126
64	130
65	129
55	111
58	116
57	112

Space for hints

Space for hints

Let x denote the variable, height and y denote the variable, weight.

Also let $A = 60$; $B = 120$

x	$(x-A)=d_x$	d_x^2	y	$(y-B)=d_y$	d_y^2	d_xd_y
57	–3	9	113	–7	49	21
59	–1	1	117	–3	9	3
62	2	4	126	6	36	12
63	3	9	126	6	36	18
64	4	16	130	10	100	40
65	5	25	129	9	81	45
55	–5	25	111	–9	81	45
58	–2	4	116	–4	16	8
57	–3	9	112	–8	64	24
Total	0	102		0	472	216

$\Sigma d_x = 0$; $\Sigma d_y = 0$;

$\Sigma d_x^2 = 102$; $\Sigma d_y^2 = 472$; $\Sigma d_xd_y = 216$

n = Total number of pairs of values of x and y given = 9

$$\therefore r = \frac{\left(n \times \Sigma d_xd_y - (\Sigma d_x)(\Sigma d_y)\right)}{\sqrt{\left[n \times \Sigma d_x^2 - (\Sigma d_x)^2\right] \times \left[n \times \Sigma d_y^2 - (\Sigma d_y)^2\right]}}$$

$$= \frac{(9 \times 216) - (0 \times 0)}{\sqrt{\left[9 \times 102 - (0)^2\right] \times \left[9 \times 472 - (0)^2\right]}}$$

$$= \frac{1944}{\sqrt{918 \times 4248}}$$

value of $\sqrt{918 \times 4248}$ is found out using the log table as follows:

$$\log 918 = 2.9628$$

$$\log 4248 = 3.6282$$

$$\log 918 + \log 4248 = 2.9628 + 3.6282$$

$$= 6.5910$$

$$\frac{\log 918 + \log 4248}{2} = \frac{6.5910}{2} = 3.2955$$

$$\text{Anti-log } (3.2955) = 1974$$

$$\sqrt{918 \times 4243} = 1974$$

$$\therefore r = \frac{1944}{1974} = 0.9849$$

$\therefore$ Correlation coefficient between the values of height and weight = .9849

15.6 Interpretation of Karl Pearson's coefficient of correlation :

The value of Pearson's correlation coefficient, r, always lies between the two values +1 and –1. When the value of r is positive it denotes positive or direct relationship between the given variables. A negative value of r denotes negative or inverse relationship between the two given variables. Now let us see the interpretation given to various numerical values*of r.

(i) When the numerical value of r is equal to 1 the correlation is perfect, and the exact value of one of the two variables may be obtained from a known value of the other.

(ii) If the numerical value of r is greater than .95, there is a high degree of correlation between the variables and one them may be quite accurately estimated from a known value of the other.

(iii) If the numerical value of r is greater than .75, but less than .85 there probably is a decided amount of correlation between two variables, and one of the variables may be estimated roughly from a known value of the other variable.

(iv) If the numerical value of r lies between .4 and .6 there may be a fair degree of association between the two variables, but any estimate of the value of one variable from a known value of the other would ordinarily be of but little practical value.

(v) If the numerical value is less than .4, there is very low degree of correlation between the two variables and the known values of the variable cannot be used as the basis for estimating the value of the other.

(vi) When the value of r is equal to zero, the two given variables have no linear relationship. That is, zero value of r indicates only the absence of linear correlation and not the absence of correlation itself.

The general values mentioned above, of course are arbitrary and for this reason gaps were left between the general zones of the values of x.

* Ignoring the sign of r if we consider the number denoting the value of r, it is the numerical value of r.

Space for hints

Space for hints

15.7 Merits of Pearson's Coefficient of Correlation:

It reveals the nature of correlation between two variables (whether positive or negative) and at the same time, it gives numerical measure of the correlation between the two variables.

15.8 Demerits:

1) Whether the correlation between the two given variables is linear or non-linear, we assume it to be linear while we calculate Pearson's Coefficient of correlation .

2) The calculation of Pearson's coefficient of correlation involves much time compared to other methods of finding out correlation .

3) Very large values and very small values present in the given series influence the value of r to a great extent. Hence, sometimes the value of r yields false conclusions.

Example -5 :

Explain the concept of coefficient of correlation and find out the r for the following data and interpret the 'r'

$N=12$ $\qquad\qquad$ $\Sigma x' = 23$

$\Sigma x'^2 = 921$ $\qquad\qquad$ $\Sigma y' = -24$

$\Sigma x'y' = -433$ $\qquad$ $\Sigma y'^2 = 734$

By way of answer to the first part of this question, the students are to give the object of coefficient of correlation (r), the assumption on which it is based, the formula to calculate r, the limits of r and the interpretation given to various values of r.

Now we come to the second part of the question. In the lesson, we have given the short-cut formula to calculate r, as follows:

$$\therefore r = \frac{\left(n \times \Sigma d_x d_y - (\Sigma d_x)(\Sigma d_y)\right)}{\sqrt{\left[n \times \Sigma d_x^2 - (\Sigma dx)^2\right] \times \left[n \times \Sigma d_y^2 - (\Sigma d_y)^2\right]}}$$

Here instead of d_x and d_y, x' and y' are used in the question to represent (x–A) and (y–B) respectively. Instead of n, N is used to represent number of

pairs of values of x and y given. So correspondingly the short cut formula becomes as follows.

Space for hints

$$r = \frac{\left(n \times \Sigma d_x d_y - (\Sigma d_x)(\Sigma d_y)\right)}{\sqrt{\left[n \times \Sigma d_x^2 - (\Sigma d_x)^2\right] \times \left[n \times \Sigma d_y^2 - (\Sigma d_y)^2\right]}}$$

Given $N = 12$, $\Sigma x'^2 = 921$ $\Sigma x'y' = -433$, $\Sigma x' = 23$, $\Sigma y' = -24$, $\Sigma y'^2 = 734$

$$r = \frac{[12 \times (-433)] - [(23) \times (-24)]}{\sqrt{\left[12 \times 921 - [23]^2\right] \times \left[12 \times 734 - [-24]^2\right]}}$$

$$= \frac{-4644}{\sqrt{10523 \times 8232}}$$

$$\log 4644 = 3.6669$$

$$\log 10523 = 4.0220$$

$$\log 8232 = 3.9155$$

$$\log 10523 + \log 8232 = 4.0220 + 3.9155$$

$$= 7.9375$$

$$\frac{\log 10523 + \log 8232}{2} = \frac{7.9375}{2} = 3.9687$$

$$\log 4644 - ½(\log 10523 + \log 8232) = 3.6669 - 3.9687$$

$$= \bar{1}.6982$$

$$\text{Anti-log } (\bar{1}.6982) = 0.4991$$

$$r = \frac{-4644}{\sqrt{10523 \times 8232}} = -.4991$$

Here the value of r is a negative fraction and hence the given two variables are negatively correlated. In other words, the given two variables are inversely related to one another.

Space for hints

16. Rank Correlation

(a) Meaning :

The relationship between two variables can be studied in the following two ways:

(1) By studying the relationship between actual values of the two variables; and,

(2) By studying the relationship between the ranks of the values of the two variables.

Consider the following table giving the marks obtained by four students in Economics and Statistics:

Economics	Statistics
61	73
50	66
55	82
63	72

We can consider the marks in Economics as a variable and that in Statistics as another variable and the marks given in the table are the actual values of these two variables. Now using Karl Pearson's Correlation Coefficient (which we have explained in the Previous lesson) we can find out whether there is any correlation between the two given variables. This is a study of the relationship between the actual values of the two variables.

The correlation between the two variables given above can also be studied in another way. We can give ranks to the values of the two variables. Ranking of the values of a variable is done as follows. The highest value of the variable is given the first rank (i.e) the rank of the highest value is 1. In our example, consider the variable viz. marks in Economics, its highest value given is 63, it is given the rank 1.

The next higher value is given the second rank (i.e) the rank of the next higher value is 2. In our example, the next higher value is 61. Therefore, its rank is 2.

The next higher value is given the third rank, In our example, the next higher value is 55. Therefore, its rank is 3.

In the same way, the rest of the values given are also ranked.

In our example, we have only one more value. Hence its rank is 4.

Similarly we can rank the values of the variable viz., marks in Statistics

also. Now we can give the ranks of the values of the two variables given in a table as follows:

Economics		Statistics	
Marks	Rank	Marks	Rank
61	2	73	2
50	4	66	4
55	3	82	1
63	1	72	3

In the above example, each value of the given variable is different from the other values of the same variable. But sometimes two or more values of the given variable may be equal. Consider the variable marks in Economics. It is possible for two or more students to get the same marks. In such cases, the method of ranking is as follows:

Marks	45	50	51	45	45	49	50

In the above table the highest mark is 51 and is got by the third student.

$\therefore$ The rank of the third student is 1.

The next higher mark is 50. It is got by the second student and by the last student. Therefore, both these, second and last students, should be given the same rank. The common rank to be assigned to both of them is found out as follows:

If the marks of the second and last students differed slightly, their ranks would be 2 and 3. We consider these two ranks and find out their arithmetic average.

$$\text{Arithmetic mean of 2 and 3} = \frac{2+3}{2} = \frac{5}{2} = 2.5$$

We take this 2.5 as the common rank and assign this rank to the second and last students.

The next higher mark is 49 and is got by the last but one student. We attach the rank 4 to this student. It is to be noted that there is no second rank and no third rank.

The next higher mark is 45. It is got by three students viz., the first, fourth and fifth students. Because all these three students have got the same mark, they should be assigned the same rank. As before we consider the ranks of these three students if their marks would have differed slightly. These ranks are 5,6 and 7. We find out the arithmetic mean of these three ranks viz, 5, 6, 7.

Space for hints

Space for hints

Arithmetic mean is equal to 6 $\left(=\frac{5+6+7}{2}=\frac{18}{2}=6\right)$

We assign the rank 6 to the first, fourth and fifth students' Now we have assigned ranks to all students given. We give below the marks and ranks as follows:

Marks	Ranks
45	6
50	2.5
51	1
45	6
45	6
49	4
50	2.5

From the above example it follows that when two or more values given are equal, we assign a common rank to all these values. This common rank is the arithmetic mean of the ranks they would have received if they had differed slightly.

We can study the relationship between two variables by studying the relationship between the ranks of the values of the two given variables also. The relationship that exists between the ranks is known as 'rank correlation'.

(b) Method of measuring rank correlation :

Spearman has given the formula to calculate the rank correlation coefficient and hence it is called 'Spearman's rank correlation coefficient'. The formula is as follows:

$$R = 1 - \frac{6\Sigma d^2}{n(n^2-1)}$$

Where R denotes the rank correlation coefficient.

n denotes the total number of pairs of ranks given.

d denotes the difference between the ranks in each pair of ranks given.

(c) The procedure to calculate the rank correlation coefficient :

1. If we are given actual values of two variables we find out the ranks of values of each variable as we have explained earlier. Now we have a series of pairs of ranks of the two variable given.
2. We find out the differences between the ranks in each pair and denote this difference by d.

Check your Progress

13. What is rank correlation?

14. Define Spearman's rank correlation co-efficient.

Space for hints

3. We find out the squares of the values of d. That, is, we find out the values of d^2. We sum up the values of d^2 and denote the sum by Σd^2.

4. We find out the total number of pairs of ranks given and denote this number by n.

5. Now using the formula $R = 1 - \frac{6\Sigma d^2}{n(n^2-1)}$ we find out the rank correlation coefficient.

Example 1 :

Marks obtained by 10 students in Mathematics and Statistics are given below. Find out the rank correlation coefficient.

Marks in Mathematics	Marks in Statistics
95	63
55	55
63	47
40	60
72	48
88	42
65	69
49	70
54	51
50	45

Actual marks obtained by 10 students in two subjects are given above. To calculate the rank correlation coefficient first of all we find out the ranks of the marks given above. We have given below the ranks in a table as follows;

Mathematics		Statistics	
Mark	Rank	Marks	Rank
95	1	63	3
55	6	55	5
63	5	47	8
40	10	60	4
72	3	48	7
88	2	42	10
65	4	69	2
49	9	70	1
54	7	51	6
50	8	45	9

Space for hints

Total number of pairs of ranks given is 10.

$\therefore$ n = 10

In the table below, we have given the ranks of marks in Mathematics under the heading x and the ranks in Statistics under the heading y. (x – y) values are denoted by by d and given in the column with heading, d. Square value of d (i.e.) d^2 also computed and given under the heading d^2.

x	y	d	d^2
1	3	(1–3) = –2	$(-2)^2 = 4$
6	5	(6–5) = 1	$(1)^2 = 1$
5	8	(5–8) = –3	$(-3)^2 = 9$
10	4	(10–4) = 6	$(6)^2 = 36$
3	7	(3–7) = –4	$(-4)^2 = 16$
2	10	(2–10) = –8	$(-8)^2 = 64$
4	2	(4–2) = 2	$(2)^2 = 4$
9	1	(9–1) = 8	$(8)^2 = 64$
7	6	(7–6) = 1	$(1)^2 = 1$
8	9	(8–9) = –1	$(1)^2 = 1$
Total			$\Sigma d^2 = 200$

Now knowing the values of n and d^2 we calculate the value of rank correlation coefficient R, using the formula.

$$R = 1 - \frac{6\Sigma d^2}{n(n^2-1)}$$

In our example, n = 10 and $\Sigma d^2 = 200$

$$\therefore R = 1 - \frac{6\times 200}{10(10^2-1)}$$

$$= 1 - \frac{6\times 20}{(100-1)}$$

$$= 1 - \frac{6\times 20}{99}$$

Space for hints

$$= 1-\frac{2\times 20}{33} = 1-\frac{40}{33}$$

$$= \frac{33-40}{33}$$

$$= \frac{-7}{33} = -.212$$

∴ Rank correlation coefficient = – .212

Example 2:

In a certain examination 6 students obtained the following marks in Mathematics and Statistics. Find out the rank correlation coefficient.

Mathematics	Statistics
90	73
82	66
77	82
78	72
72	51
69	34

We are given the actual marks obtained by the six students and we are asked to find out the rank correlation coefficient. So first of all we find out the rank of each student in each subject and then proceed to calculate the rank correlation coefficient.

We give the ranks of the 6 students in Mathematics under the heading x and their ranks in Statistics under the heading y in the table below.

Total number of pairs of ranks given is 6.

∴ n = 6.

We find out the difference between the ranks in each pair and give it under the heading d in the table below:

Squares of the values of d are found out and given under the heading d^2

Space for hints

in the table below.

x	y	d	d^2
1	2	$(1-2) = -1$	$(-1)^2 = 1$
2	4	$(2-4) = -2$	$(-2)^2 = 4$
3	1	$(3-1) = 2$	$(2)^2 = 4$
4	3	$(4-3) = 1$	$(1)^2 = 1$
5	5	$(5-5) = 0$	$(0)^2 = 0$
6	6	$(6-6) = 0$	$(0)^2 = 0$
Total			10

$n = 6$ and $\Sigma d^2 = 10$

$$\therefore R = 1 - \frac{6\Sigma d^2}{n(n^2-1)}$$

$$= 1 - \frac{6 \times 10}{6(6^2-1)}$$

$$= 1 - \frac{10}{(36-1)}$$

$$= 1 - \frac{10}{35}$$

$$= 1 - \frac{2}{7}$$

$$= \frac{7-2}{7}$$

$$= \frac{5}{7}$$

$$= .714$$

$\therefore$ Rank correlation coefficient = .714.

Example 3 :

Space for hints

Percentage of marks obtained by 8 Students in Economics and Statistics are as follows:

Economics	Statistics
80	82
45	56
55	50
58	48
55	56
60	62
45	64
68	65

Calculate the rank correlation coefficient.

The mark of the first student is 80 and it is highest mark.

∴ The rank of first student is 1.

68 is the next higher mark and is got by the last student.

∴ The rank of the last student is 2.

60 is the next to 68 and the sixth student has got this mark.

∴ The rank of the sixth student is 3.

58 is the next mark which is the mark of the fourth student.

∴ The rank of the fourth student is 4.

55 is the next mark which has been obtained by both the third and fifth student, if their marks has differed slightly their ranks would be 5 and 6.

∴ We assign the rank $\frac{5+6}{2} = \frac{11}{2} =$ 5.5 to both the students who have got the same marks viz., 5.5

45 is the next mark. It has also been obtained by two students viz., the second and seventh students. If the marks of these two students had differed slightly, their ranks, will be 7 and 8

Space for hints

∴ We assign the $\frac{7+8}{2}=\frac{15}{2}=7.5$ rank to both the students who have got the same marks, viz., 45.

In the same way we find out the rank of the eight students in statistics.

The first student has got the highest mark viz., 82 and hence his ranks is 1.

The last student has got the next higher marks viz., 65 and hence his rank is 2.

Seventh student has got the next higher marks viz., 64 and hence his rank is 3.

Sixth student has got the next higher marks viz., 62 and hence his rank is 4.

Second as well as the fifth student have got the next mark viz., 56.

Both of them must be given the same rank $= \frac{5+6}{2}=\frac{11}{2}=5.5$

Third student has got the next mark viz., 50 and hence his rank is 7.

Fourth student has got the last mark viz., 48 and his rank is 8.

Now we give the ranks which we have found out in a table as follows:

Economics		Statistics	
Mark	Rank	Marks	Rank
80	1	82	1
45	7.5	56	5.5
55	5.5	50	7
58	4	48	8
55	5.5	56	5.5
60	3	62	4
45	7.5	64	3
68	2	65	2

Having found out the ranks we calculate the rank correlation coefficient as usual using the formula:

$$R = 1 - \frac{6\sum d^2}{n(n^2-1)}$$

We have given the ranks in Economics under the heading x and the ranks in Statistics under the heading y in the table below:

Space for hints

x	y	d	d^2
1	1	(1–1) = 0	$(0)^2$ = 0
7.5	5.5	(7.5–5.5) = 2	$(2)^2$ = 4
5.5	7	(5.5–7) = –1.5	$(-1.5)^2$ = 2.25
4	8	(4–8) = –4	$(-4)^2$ = 16
5.5	5.5	(5.5–5.5) = 0	$(0)^2$ = 0
3	4	(3–4) = –1	$(-1)^2$ = 1
7.5	3	(7.5–3) = 4.5	$(4.5)^2$ = 20.25
2	2	(2 –2) = 0	$(0)^2$ = 0
Total			= 43.50

$$\Sigma d^2 = 43.50$$

$$n = \text{Total number of ranks}$$

$$= 8$$

$$\therefore R = 1 - \frac{6\sum d^2}{n(n^2-1)}$$

$$= 1 - \frac{6 \times 43.5}{8(8^2-1)}$$

$$= 1 - \frac{3 \times 43.5}{4(64-1)}$$

$$= 1 - \frac{3 \times 43.5}{4 \times 63}$$

$$= 1 - \frac{43.5}{4 \times 21}$$

$$= 1 - \frac{14.5}{4 \times 7}$$

$$= \frac{28 - 14.5}{28}$$

Space for hints

$$= \frac{13.5}{28}$$

$$= .48$$

Rank correlation coefficient = .48

(d) Application of Rank Correlation Coefficient

1. When the two variables given are qualitative variables it is possible only to arrange the values of the two variables in a rank order and we cannot express the values in quantitative terms. For example, if the given variable is intelligence, we cannot measure it numerically. We can only rank the persons according to their intelligence. In such cases, correlation between two given variables can be studied only through rank correlation coefficient. Karl Pearson's coefficient of correlation cannot be used.

Sometimes, even when the given two variables are quantitative variables. Spearman's rank correlation coefficient is used because it gives a more accurate measure of correlation than Karl Pearson's coefficient of correlation.

2. Sometimes we may be given more than two sets of ranks assigned by different people to the values of the same variable and we may be asked to find out the pair of people who have common tastes. For example, three or more judges may be employed and asked to rank the given of set ladies according to their beauty or to rank the given set of students according to their intelligence. Different judges may assign different ranks to the same person. Therefore we will have more than two sets of ranks. With the help of these sets of ranks the pair of people who have common tastes is found out as follows:

We consider the given sets of ranks pair–wise (i.e.,) if we are given three sets, we consider the first and second sets together, second and third sets together, and first and third sets together. We find out the rank correlation coefficient of each pair of sets of ranks. Then the pair having highest positive rank correlation coefficient is found out. The judges corresponding to this selected pair of sets of ranks are said to have common tastes. The following worked out example will explain the things given above.

Example 4 :

Ten competitors in a voice test are ranked by three judges as shown in the following table:

Judge I	Judge II	Judge III
1	3	6
6	5	4
5	8	9
10	4	8
3	7	1
2	10	2
4	2	3
9	1	10
7	6	5
8	9	7

Space for hints

Using the method of rank correlation find out, which pair of judges has the nearest approach to likings in voice.

In this problem we find out the rank correlation coefficient of I and II, I, III and II and III separately using the formula,

$$R = 1 - \frac{6\Sigma d^2}{n(n^2 - 1)}$$

The correlation coefficient which is the highest and positive gives the pair of judges who has the nearest approach to likings in voice.

First we find out the rank correlation coefficient to I and II as follows:

I	II	d	d^2
1	3	$(1-3) = -2$	$(-2)^2 = 4$
6	5	$(6-5) = 1$	$(1)^2 = 1$
5	8	$(5-8) = -3$	$(-3)^2 = 9$
10	4	$(10-4) = 6$	$(6)^2 = 36$
3	7	$(3-7) = -4$	$(-4)^2 = 16$
2	10	$(2-10) = -8$	$(-8)^2 = 64$
4	2	$(4-2) = 2$	$(2)^2 = 4$
9	1	$(9-1) = 8$	$(8)^2 = 64$
7	6	$(7-6) = 1$	$(1)^2 = 1$
8	9	$(8-9) = -1$	$(-1)^2 = 1$
Total			200

Space for hints

$$\Sigma d^2 = 200$$

Here, n = Total number of pairs of ranks

$$= 10$$

$$\therefore R = 1 - \frac{6\Sigma d^2}{n(n^2-1)}$$

$$= 1 - \frac{6 \times 200}{10\,(100-1)}$$

$$= 1 - \frac{6 \times 20}{99}$$

$$= 1 - \frac{120}{99}$$

$$= 1 - 1.212 = -.212$$

Rank correlation coefficient of I and III is found out as follows:

I	II	d	d^2
1	6	$(1-6) = -5$	$(-5)^2 = 25$
6	4	$(6-4) = 2$	$(2)^2 = 4$
5	9	$(5-9) = -4$	$(-4)^2 = 16$
10	8	$(10-8) = 2$	$(2)^2 = 4$
3	1	$(3-1) = 2$	$(2)^2 = 4$
2	2	$(2-2) = 0$	$(0)^2 = 0$
4	3	$(4-3) = 1$	$(1)^2 = 1$
9	10	$(9-10) = -1$	$(-1)^2 = 1$
7	5	$(7-5) = 2$	$(2)^2 = 4$
8	7	$(8-7) = 1$	$(1)^2 = 1$
Total			60

$$\Sigma d^2 = 60$$

$$n = 10$$

Space for hints

$$\therefore R = 1 - \frac{6\Sigma d^2}{n(n^2-1)}$$

$$= 1 - \frac{6 \times 60}{10(10^2-1)}$$

$$= 0.636$$

Rank correlation coefficient of II and III is found out as follows:

II	III	d	d^2
3	6	$(3-6) = -3$	$(-3)^2 = 9$
5	4	$(5-4) = 1$	$(-1)^2 = 1$
8	9	$(8-9) = -1$	$(-1)^2 = 1$
4	8	$(4-8) = -4$	$(-4)^2 = 16$
7	1	$(7-1) = 6$	$(6)^2 = 36$
10	2	$(10-2) = 8$	$(8)^2 = 64$
2	3	$(2-3) = -1$	$(-1)^2 = 1$
1	10	$(1-10) = -9$	$(-9)^2 = 81$
6	5	$(6-5) = 1$	$(1)^2 = 1$
9	7	$(9-7) = 2$	$(2)^2 = 4$
Total			214

$$\Sigma d^2 = 214$$

$$n = 10$$

$$\therefore R = 1 - \frac{6\Sigma d^2}{n(n^2-1)}$$

$$= 1 - \frac{6 \times 214}{10(10^2-1)}$$

$$= 1 - \frac{6 \times 214}{10 \times 99}$$

Space for hints

$$= 1 - \frac{2 \times 107}{5 \times 33}$$

$$= 1 - \frac{214}{165}$$

$$= \frac{165 - 214}{165}$$

$$= \frac{-49}{165}$$

$$= -.297$$

∴ Rank correlation coefficient of I and II = – .212

Rank correlation coefficient of I and III = +.636

Rank correlation coefficient of II and III = –.297

Only rank correlation coefficient of I and III is positive and the other two are negative.

∴ Judges I and III have the nearest approach to likings in voice.

(e) Interpretation of Spearman's Rank Correlation Coefficient :

(i) Rank correlation coefficient also always lies between +1 and –1

(ii) When R = +1, there is complete agreement in the order of ranks and the ranks are in the same directions.

(iii) When R= –1 there is complete disagreement in the order of ranks and the ranks are in the opposite directions.

(iv) When R is a positive fraction there is agreement to a limited degree in the same direction.

(v) When R is a negative fraction there is agreement to a limited degree in the opposite direction.

(vi) When R is zero, there is no agreement at all in the order of ranks.

17. Usefulness of the study of Correlation Analysis

1. Study of the relation between supply and price and between demand and price is very helpful in forecasting the price of commodities.

Space for hints

2. Analysis of correlation between advertising expenditure and sales is helpful to improve the sales prospects. Analysis of the correlation between sales (or production) and certain characteristics of salesmen (or workers) such as education special training, ages etc., helps the management in avoiding waste of time, effort and finance.

3. The correlation analysis of sales with personal income is useful in market analysis.

4. Correlation coefficients are applied in the development and application of mental tests. They are used for measuring various types of mental abilities such as general intelligence or aptitude for special pursuits for measuring achievement in a defined course of study and for predicting degree of success in carrying out certain tasks.

5. The correlation between income and expenditure on food and the correlation between the expenditure on food and the distribution of expenditure on different types of food are of great value (i) in planning (ii) research on family nutrition habits and (iii) in setting up agricultural marketing programmes.

6. The correlation coefficient is used in personnel selection.

7. Application of correlation methods in the fields or sociology and political science is increasing day by day. The sociologist may be interested in the relationship between unemployment and poverty. A politician may be interested in the relationship between the composition of population and proportion of votes cast for a particular party; or he may be interested in the relationship between the education of individuals and their political preferences.

18. Answers to Check Your Progress Questions :

1.	Refer 1	8.	Refer 7.1
2.	Refer 2(i)	9.	Refer 7.3
3.	Refer 2(ii)	10.	Refer 9.2
4.	Refer 5.1	11.	Refer 13
5.	Refer 5.2	12.	Refer 15
6.	Refer 6.1	13.	Refer 16(a)
7.	Refer 6.2	14.	Refer 16(b)

Space for hints

19. Model questions for guidance :

10 Marks Questions (One Page Answer)

1. What is meant by correlation between two variables? Explain the concept with suitable examples.
2. What is meant by positive and negative correlation?
3. Write shorts notes on Rank Correlation.
4. Define correlation and explain briefly its uses.
5. Mention the posssible values of r and interpret them.

20 Marks Questions (Three Page Answer)

1. Write short notes on:

   (i) Scatter diagram (ii) Correlation graph

2. Calculate Spearman's correlation coefficient for the following:

x	28	41	40	38	35	33	40	32	36	33
y	23	34	33	34	30	26	28	31	35	38

   What inference would you draw from the estimate?

3. Calculate the Rank correlation coefficient between x and y

x	78	89	97	69	59	79	68	57
y	125	137	156	112	107	136	123	108

4. Ten competitors in a beauty contest are ranked by three Judges in the order as given below:

Judge I	5	7	1	3	10	2	6	4	8	9
Judge II	1	5	3	4	7	6	10	9	2	8
Judge III	4	6	8	9	2	1	3	7	10	5

   Determine which pair of judges have the nearest approach to common tastes in beauty.

5. Calculate Pearson's correlation coefficient for the following

X	28	27	28	29	30	31	33	35	36	39
Y	18	32	23	24	25	26	28	29	30	32

UNIT – 5

REGRESSION

Space for hints

Introduction

In Unit-4 we described the method of finding out whether there is any relationship between the variables namely correlation. When we are sure that there exists relationship between two or more variables, we may be interested in knowing the unknown value of one variable given the value of the related variable. To fulfil this purpose, there is a statistical tool known as 'Regression'. We explain both Regression in this Unit-5.

Unit Objectives :

After studying this Unit, you would be able to understand

* Meaning of regression, regression lines and regression equations
* Estimation of regression equations
* Meaning and properties of regression coefficients.

Unit Structure :

1. Regression - Meaning
2. Purpose of regression analysis
3 Regression Lines
4 Properties of Regression Lines
5 Regression Equations
6 Procedure to get the Regression Equations
7. Procedure to get the unknown value of one variable given a value of the other variables
8. Regression Coefficients
9. Properties of regression coefficients
10. Utility or importance of regression analysis
11. Difference between correlation and regression
12. Answers to Check Your Progress Questions
13. Model questions for guidance

1. Regression - Meaning :

Regression literally means the tendency to return. The word 'Regression' was used by a well-known statistician, Sir Francis Galton in his studies connected with the relation between the height of fathers and the height of sons. He observed that on the average the off-spring of tall fathers were not so tall as their fathers, while the off spring of short fathers were not so short as their

Check your Progress

1. What is regression?

Space for hints

fathers. The sons tended to go back or regress towards the average height of the race. He called this return towards average the 'regression'. But now this term is not used in the same sense as envisaged by Galton. In statistics, 'regression' means simply average relationship between variables.

2. Purpose of Regression Analysis :

The main purpose of regression analysis is to predict or estimate the unknown values of one variable from the known values of another related variable. For example, the producer of a commodity may wish to estimate the amount of rupees to be spent on advertisement to have a particular amount of sales; or he may wish to estimate the sales when a particular amount of rupees is spent on advertisement. Such type of estimation is possible with the help of regression analysis.

Regression can be expressed graphically or algebraically. Graphic representation of regression is called 'regression' lines. Regression expressed algebraically is called 'regression equation'*

3 Regression Lines:

The average relationship between two variables is described by the regression lines. In other words, when the exact value of one variable is given the most probable value of the other variable is shown by the regression line.

When two related variables are given, there are two regression lines. Suppose, x and y are two related variables. One regression line will give the best estimate of the values of x when the values of y are given. And this is called the **'regression line of x on y'**. The other regression line will give the best estimate of the values of y when the values of x are given. This line is called **regression line of y on x.**

4 Properties of Regression Lines :

(i) When r = ±1

If the correlation between the two given variables is either perfectly positive ($r = +1$) or perfectly negative ($r = -1$), the two regression lines will coincide with each other. That is, in such a case we will have only one regression line instead of two regression lines as shown in fig.5.1 and 5.2.

Check your Progress

2. What are regression lines?

3. How many regression lines will be there when $r = \pm 1$?

---

* Sometimes the algebric expression is also called by the same name 'regression line'.

Fig. 5.1
Regression Line when r = +1

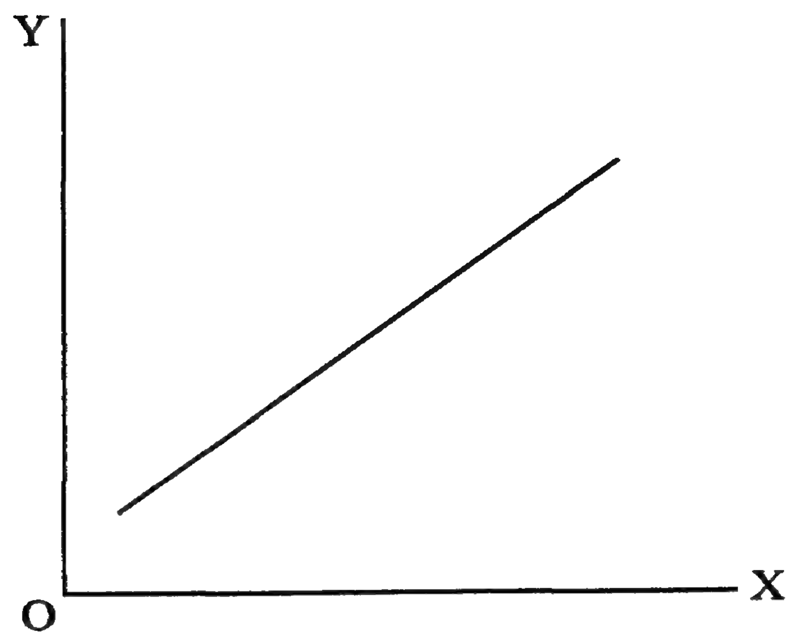
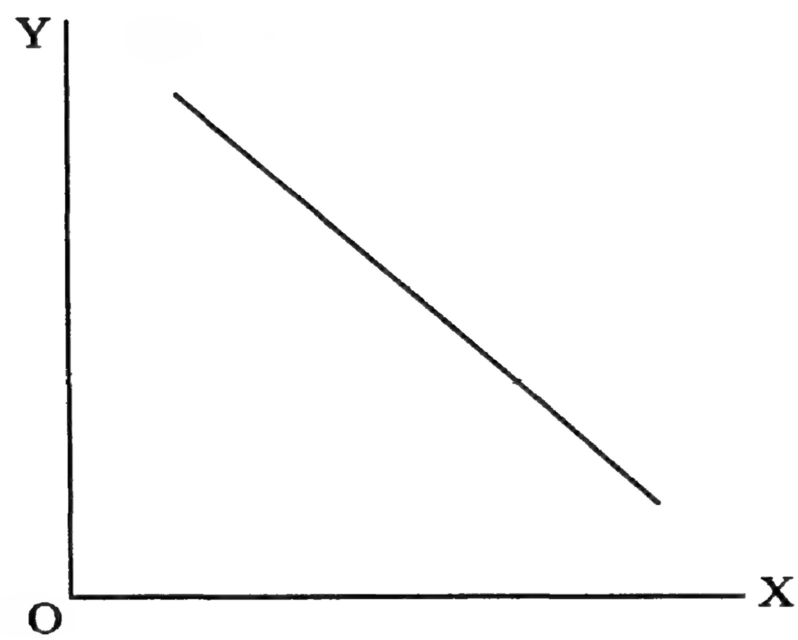


Fig. 5.2
Regression Line when r = – 1



(ii) When r is very close to ±1

If the degree of correlation between the two variables is higher. (i.e. when the value of r is very nearer to +1 or –1) the two regression lines will be nearer to each other.

(iii) When r is very near to zero

On the other hand, if the degree of correlation between the two variables is smaller (i.e. when the value of r is nearer to 0) the two regression lines will be farther away from each other. The angle between the two lines will show whether the two lines are nearer or farther away from each other.

The two regression lines in fig. 5.3 and 5.4 represent high degree of correlation as they are closer to each ohter.

Space for hints

Space for hints

Fig. 5.3

Regression Lines when r is nearer to +1

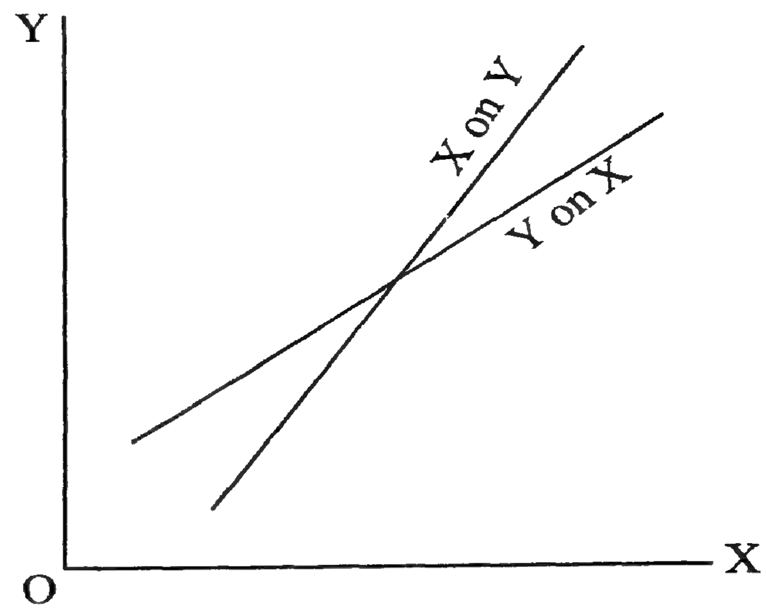


Fig. 5.4

Regression Lines when r is nearer to −1

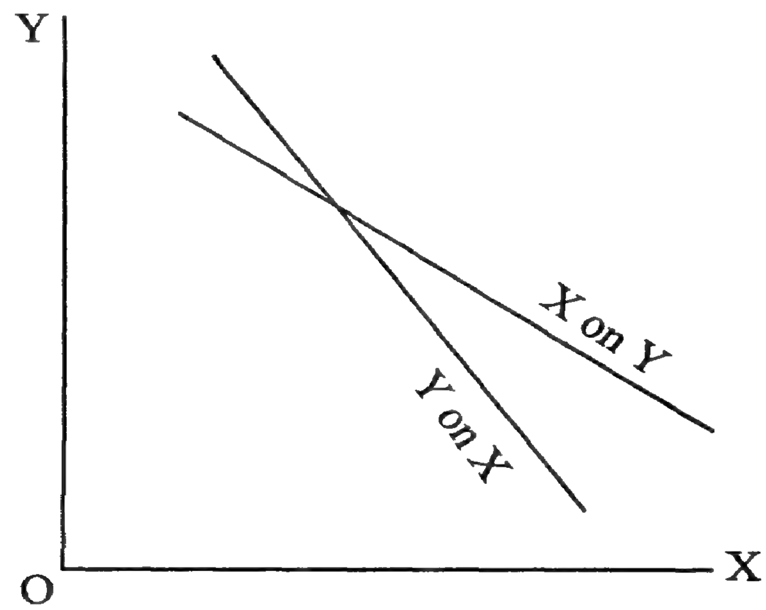
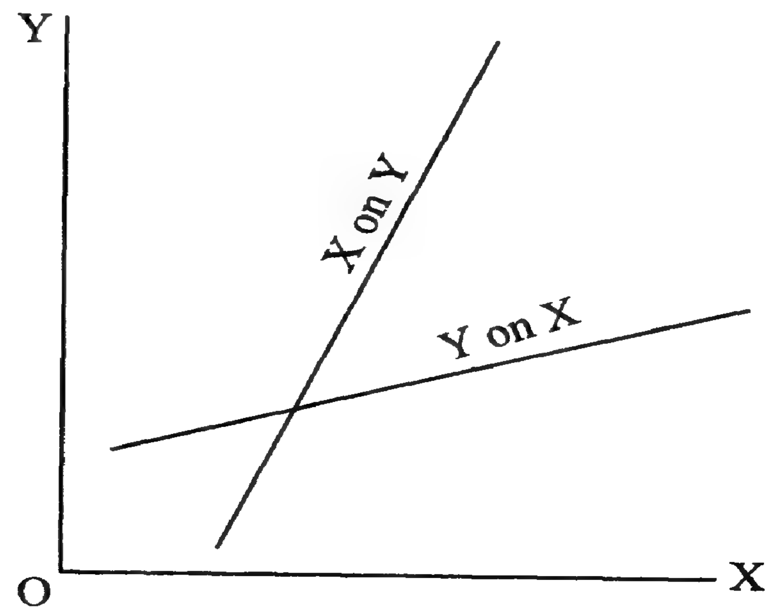


Fig. 5.5

Regression Lines when r is nearer to zero



Space for hints

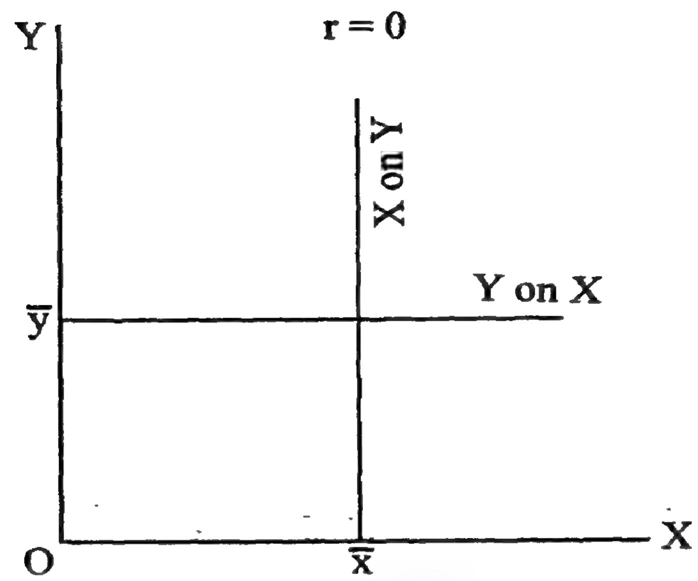
The two regression lines in fig. 5.5 are further away from each other representing low of degree of correlation.

(iv) When r = 0

When there is no correlation between the two variables (i.e.) when the value of r = 0, the regression lines will intersect each other at right angles as follows:

Fig. 5.6

Regression Line when r = 0



(iv) Point of intersection of the two regression lines

The two regression lines will intersect each other at the point $(\bar{x},\bar{y})$.

5 Regression Equations

Regression lines are often given algebraical expressions. Algebraical expression of a regression line is called a 'regression' equation. Regression equations help us to draw the regression lines and they provide us a numerical method of finding out the estimated value of one variable form the values of the other variable. We have already said that when we are given a series of values of two variables we will have two regression lines. Each regression line will have a regression equation. Therefore, we have to regression equation. They are as follows:

Regression Equation of x and y :

$$(x-\bar{x}) = \frac{r\sigma x}{\sigma y}\ (y-\bar{y}) \qquad \text{..............(1)}$$

Regression equation of y on x :

$$(y-\bar{y}) = \frac{r\sigma y}{\sigma x}\ (x-\bar{x}) \qquad \text{...............(2)}$$

Space for hints

In the above two equations, $\bar{x}$ is the mean of the x values and $\bar{y}$ is the mean of the y values in the given series. σx and σy are the standard deviations of the x values and y values given respectively; r is the correlation coefficient between x and y.

If a value of y is given and if we are asked to estimate the value of x, we should use the equation (1) only.

If a value of x is given and if we are asked to estimate the value of y, we should use the equation (2) only.

6 Procedure to get the two Regression Equations

Given a series of values of two variables x and y:

(i) First find out the mean of the values of x in the series and denote it by $\bar{x}$. Then find out the mean of y in the series and denote this mean value by $\bar{y}$.

(ii) Find out the standard deviation of the values of x and denote it by σx. Find out the standard deviation of the values of y given and denote by σy.

(iii) Find out correlation coefficient (r) between x and y.

Now knowing the values of $\bar{x}$, $\bar{y}$, σx, σy and r we can get the two regression equations.

$$(x-\bar{x}) = \frac{r\sigma x}{\sigma y}\ (y-\bar{y})$$

$$(y-\bar{y}) = \frac{r\sigma y}{\sigma x}\ (x-\bar{x})$$

Example 1:

Pairs of values of two variables x and y are given below. Get the two regression equations.

x	y
78	125
89	137
97	156
69	112
59	107
79	136
68	123
61	104

First we find out the mean of the x values ($\bar{x}$) and the mean of the y values ($\bar{y}$)

Space for hints

$$\bar{x} = \frac{\Sigma x}{n} = \frac{600}{8} = 75$$

$$\bar{y} = \frac{\Sigma y}{n} = \frac{1000}{8} = 125$$

We find out the deviation of each value of x from $\bar{x}$ and give it under the heading $(x-\bar{x})$ in the table below. We find out the squares of the values of $(x-\bar{x})$ (i.e) the values of $(x-\bar{x})^2$ and give them under the heading $(x-\bar{x})^2$.

Now using the formula,

$$\sigma_x = \sqrt{\frac{\Sigma(x-\bar{x})^2}{n}}$$

We find out the standard deviation of value of x,

In the same way, the deviation of each value of y from $\bar{y}$ is found out and given under the heading $(y-\bar{y})$ in the table below. We find out the squares of the values of $(y-\bar{y})$ (i.e) the values of $(y-\bar{y})^2$ and give them under the heading $(y-\bar{y})^2$.

Now using the formula,

$$\sigma_y = \sqrt{\frac{\Sigma(y-\bar{y})^2}{n}}$$

we find out the standard deviation of values of y.

Each value of $(x-\bar{x})$ calculated above is multiplied by the corresponding value of $(y-\bar{y})$ and given under the heading $(x-\bar{x})(y-\bar{y})$

Now we find out the value of r using the formula,

$$r = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{n\sigma_x\sigma_y}$$

$\bar{x} = 75$ $\bar{y} = 125$

Space for hints

x	$(x-\bar{x})$	$(x-\bar{x})^2$	y	$(y-\bar{y})$	$(y-\bar{y})^2$	$(x-\bar{x})(y-\bar{y})$
78	3	9	125	0	0	0
89	14	196	137	12	144	168
97	22	484	156	31	961	682
69	-6	36	112	-13	169	78
59	-16	256	107	-18	324	288
79	4	16	136	11	121	44
68	-7	49	123	-2	4	14
61	-14	196	104	-21	441	294
Total		1242			2164	1568

$$\Sigma(x-\bar{x})^2 = 1242$$

$$\Sigma(y-\bar{y})^2 = 2164$$

$$\Sigma(x-\bar{x})(y-\bar{y}) = 1568$$

$$n = 8$$

$$\sigma_x = \sqrt{\frac{\Sigma(x-\bar{x})^2}{n}}$$

$$= \sqrt{\frac{1242}{8}}$$

$$= \sqrt{\frac{621}{4}}$$

$$= \frac{\sqrt{621}}{2}$$

Value of $\sqrt{621}$ is found out as follows:

$$\log 621 = 2.7931$$

$$= \frac{\log 621}{2}$$

$$= \frac{2.7931}{2} = 1.3965$$

Space for hints

$$\text{Anti-log } 1.3965 = 24.92$$

$$\sqrt{621} = 24.92$$

$$\sigma_x = \frac{\sqrt{621}}{2} = \frac{24.92}{2} = 12.46$$

$$\sigma_y = \sqrt{\frac{\Sigma(y-\bar{y})^2}{n}}$$

$$= \sqrt{\frac{2164}{8}} = \sqrt{270.5}$$

Value of $\sqrt{270.5}$ is found out as follows:

$$\log 270.5 = 2.4322$$

$$\frac{\log 270.5}{2} = \frac{2.4322}{2} = 1.2161$$

$$\text{Anti-log}(1.2161) = 16.44$$

$$\therefore \sqrt{270.5} = 16.44$$

$$\therefore \quad \sigma_y = \sqrt{270.5} = 16.44$$

$$r = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{n\sigma_x\sigma_y}$$

$$= \frac{1568}{8\times12.46\times16.44}$$

$$= \frac{196}{12.46\times16.44}$$

$$= \frac{49}{12.46\times4.11}$$

$$= \frac{7}{1.78\times4.11}$$

$$= 0.9570$$

Space for hints

Now we give below all the values calculated above.

$$\bar{x} = 75$$

$$\bar{y} = 125$$

$\sigma_x = 12.46 \quad \sigma_y = 16.44 \quad r = .957$

$\therefore$ The regression equations of x on y is

$$(x-\bar{x}) = \frac{r\sigma x}{\sigma y}(y-\bar{y})$$

$$(x-75) = \frac{.957\times 12.46}{16.44}(y-125)$$

$$(x-75) = .7252\,(y-125)$$

$$(x-75) = .7252y - .7252\times 125$$

$$x = .7252y - 90.65+75$$

$$x = .7252y-15.65$$

$\therefore$ The regression equation of y on x is

$$(y-\bar{y}) = \frac{r\sigma y}{\sigma x}(x-\bar{x})$$

$$(y-125) = \frac{.957\times 16.44}{12.46}(x-75)$$

$$(y-125) = 1.263(x-75)$$

$$= 1.263x-1.263\times 75$$

$$= 1.263x - 94.73$$

$$\therefore y = 1.263x - 94.73 + 125$$

$$= 1.263x + 30.27$$

Answer :

The two regression equations are

$x = .7252y - 15.65$

$y = 1.263x + 30.27$

7. Procedure to get the unknown value of one variable given a value of the other variable

Space for hints

Sometimes you may be given a series of two variables x and y and you may be asked to get the two regression equations, In addition to that you may be given a value of y and asked to get the best estimate of the value of x, or you may be given a value of x and asked to get the best estimate of the value of y. In such cases first of all get the two regression equations as we have explained above. Then, in the regression equation of x on y, put the given value of y and get the value of x. It is the best estimate of the value of x. In the regression equation of y on x, put the value of x and get the value of y. It is the best estimate of the value of y.

Consider the following example:

Example - 2:

The heights (in cms) of a group of fathers and sons are given below:

Height of Father (in cms.)	Height of son (in cms.)
158	183
160	158
163	167
165	170
167	160
170	180
167	170

Find out the two regression equations and estimate the height of the son when the height of the father is 164 cms.

In this example let us consider the heights of father given to be the values of variable x.

$\therefore$ Mean of the values of X viz., $\bar{x}$ is as follows:

$$\bar{x} = \frac{158+160+163+165+167+170+167}{7}$$

$$= \frac{1150}{7} = 164.3 \text{ (approx)}$$

Space for hints

Let us consider the heights of son given to be the values of variable y.

$\therefore$ Mean of the values of Y viz. $\bar{y}$ is as follows:

$$\bar{y} = \frac{183+158+167+170+160+180+170}{7}$$

$$= \frac{1188}{7} = 169.7 \text{ (approx.)}$$

Now we form the following table to find out the values of $\sigma x, \sigma y$ and r

$\bar{x} = 164.3$ $\bar{y} = 169.7$

	x	$(x-\bar{x})$	$(x-\bar{x})^2$	y	$(y-\bar{y})$	$(y-\bar{y})^2$	$(x-\bar{x})(y-\bar{y})$
	158	-6.3	39.49	183	13.3	176.89	-83.79
	160	-4.3	18.49	158	-11.7	136.89	50.31
	163	-1.3	1.69	167	-2.7	7.29	3.51
	165	.7	.49	170	.3	.09	.21
	167	2.7	7.29	160	-9.7	94.09	-26.19
	170	5.7	32.49	180	10.3	106.09	58.71
	167	2.7	7.29	170	.3	.09	.81
Total			107.43			521.43	3.57

$$\Sigma(x-\bar{x})^2 = 107.43$$

$$\Sigma(y-\bar{y})^2 = 521.43$$

$$\Sigma(x-\bar{x})(y-\bar{y}) = 3.57$$

n = Number of values of x and y given = 7

$$\therefore \sigma_x = \sqrt{\frac{\Sigma(x-\bar{x})^2}{n}}$$

$$= \sqrt{\frac{107.43}{7}}$$

value of $\sqrt{\frac{107.43}{7}}$ is found out as follows:

$$\log 107.43 = 2.0311$$

$$\log 7 = .8451$$

$$\log 107.43 - \log 7 = 2.0311 - .8451 = 1.1869$$

$$\frac{1}{2}(\log 107.43 - \log 7) = \frac{1.1869}{2} = .5939$$

$$\text{Anti-log}\ (.5939) = 3.917$$

$$\therefore \sigma_x = \sqrt{\frac{107.43}{7}} = 3.916$$

$$\sigma_y = \sqrt{\frac{\Sigma(y-\bar{y})^2}{n}} = \sqrt{\frac{521.43}{7}}$$

Value of $\sqrt{\frac{521.43}{7}}$ is found out as follows:

$$\log 521.43 = 2.7171$$

$$\log 7 = .8451$$

$$\log 521.43 - \log 7 = 2.7171 - .8451 = 1.8720$$

$$\frac{1}{2}(\log 107.43 - \log 7) = \frac{1.8720}{2} = .9360$$

$$\text{Anti-log}\ .9360 = 8.630$$

$$\therefore \sqrt{\frac{521.43}{7}} = 8.63$$

$$\therefore \sigma_y = \sqrt{\frac{521.43}{7}}$$

$$= 8.63$$

$$r = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{n\sigma_x\sigma_y}$$

$$= \frac{3.57}{7 \times 3.917 \times 8.63}$$

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$$= \frac{.51}{3.917 \times 8.63}$$

$$= .01509$$

$$\text{Now } \bar{x} = 164.03$$

$$\bar{y} = 169.7$$

$$\sigma_x = 3.916$$

$$\sigma_y = 8.63$$

$$r = .01509$$

The regression equation of y on x is

$$(y-\bar{y}) = \frac{r\sigma_y}{\sigma_x}(x-\bar{x})$$

$$(y-169.7) = \frac{.01509 \times 8.63}{3.917}(x-164.3)$$

$$= .03325(x-164.3)$$

$$= .03325x - .03325 \times 164.3$$

$$= .03325x - 5.463$$

$$\therefore y = .03325x - 5.463 + 169.7$$

$$= .03325x + 164.237$$

The regression equations of x on y is

$$(x-\bar{x}) = \frac{r\sigma_x}{\sigma_y}(y-\bar{y})$$

$$(x-164.3) = \frac{.01509 \times 3.917}{8.63}(y-169.7)$$

$$= .006849(y-169.7)$$

$$= .006849y - (.006848 \times 169.7)$$

$$= .006849y - 1.162$$

$$\therefore x = .006849y - 1.162 + 164.3$$

$$= .006849y + 163.138$$

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We are given the height of father to be 164 cms and we are asked to find out the height of son (i.e) we are given the value of x to be 164 cms and we are asked to find out the value of y.

To find out the value of y we should use regression equation of y on x.

The regression equation of y on x is

$$y = .03325x + 164.237$$

In this equation we put the value of x to be 164

$$\therefore y = .03325 \times 164 + 164.237$$

$$= 5.453+164.237 = 169.69$$

$\therefore$ The height of the son = 169.69 cms.

Sometimes we may be given the values of $\overline{x}$, $\overline{y}$, σx, σy and r and asked to write down the two regression equations. Also we may be given a value of x and asked to estimate the value of y, and vice versa. In such cases, first of all we find out the two regression equations and then get the required estimate using the respective equation.

Example 3:

The means of variables, their standard deviations and coefficient of correlation between them are as follows:

$\overline{x} = 10, \quad \overline{y} = 20, \quad \sigma x = 1.5, \quad \sigma y = 2, \quad r = .6$

Write down the two regression equations and obtain the best estimate of x when y = 8 and the best estimate of y when x = 5.

$\overline{x} = 10, \quad \overline{y} = 20, \quad \sigma x = 1.5, \quad \sigma y = 2, \quad r = .6$

The regression equation of x on y is,

$$(x-\overline{x}) = \frac{r\sigma x}{\sigma y}(y-\overline{y})$$

$$(x-10) = \frac{.6\times 1.5}{2}(y-20)$$

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$$= .3\times1.5(y-20)$$

$$= .45(y-20)$$

$$= .45y - .45\times 20$$

$$= .45y - 9.00$$

$$\therefore x = .45y-9+10 = .45y+1$$

The regression equation of y on x is

$$(y-\bar{y}) = \frac{r\sigma y}{\sigma x}(x-\bar{x})$$

$$(y-20) = \frac{.6\times2}{1.5}(x-10)$$

$$(y-20) = .4\times2\ (x-10)$$

$$= .8(x-10)$$

$$= .8x-.8\times10$$

$$= .8x-8$$

$$y = .8x-8+20$$

$$= .8x+12$$

To get the value of x when y = 8, we should use the regression equation of x on y.

$$\therefore x = .45y + 1$$

$$= .45\times8 + 1$$

$$= 3.60 + 1$$

$$= 4.6$$

To get the value of y when x = 5, we should use the regression equation of y on x.

$$\therefore y = .8x + 12$$

$$= .8\times5 + 12$$

$$= 4.0 + 12$$

$$= 16$$

Answer :

The two regression equations are:

$$x = .45y + 1$$

$$y = .8x + 12$$

$$x = 4.6 \text{ when } y = 8$$

$$y = 16 \text{ when } x = 5$$

Example : 4

Given that the means of x and y are 65 and 67, their standard deviations are 2.5 and 3.5 respectively and the coefficient of correlation between them is .8

(a) Write down the two regressions equations.

(b) Obtain the best estimate of x when y = 70

(c) Using the estimated value of x as the given value of x estimate the value of y.

$\bar{x} = 65$, $\bar{y} = 67$, $\sigma x = 2.5$, $\sigma y = 3.5$, $r = .8$

The regression equation of x on y is,

$$(x-\bar{x}) = \frac{r\sigma x}{\sigma y}(y-\bar{y})$$

$$(x-65) = \frac{.8\times2.5}{3.5}(y-67)$$

$$= \frac{.8\times5}{7}(y-67)$$

$$= \frac{4.0}{7}(y-67)$$

$$= .57(y-67)$$

$$= .57y - .57\times67$$

$$= .57y - 38.19$$

$$\therefore x = .57y-38.19+65$$

$$= .57y+26.81$$

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The regression equation of y on x is

$$(y-\bar{y}) = \frac{r\sigma y}{\sigma x}(x-\bar{x})$$

$$(y-67) = \frac{.8\times3.5}{2.5}(x-65)$$

$$= \frac{.8\times.7}{.5}(x-65)$$

$$= \frac{.56}{.5}(x-65)$$

$$= 1.12(x-65)$$

$$= 1.12x - 1.12\times65$$

$$= 1.12x - 72.8$$

$$\therefore y = 1.12x - 72.8 + 67$$

$$= 1.12x - 5.8$$

The value of y is given as 70. To estimate the value of x we should use the regression of x on y.

$$x = .57y + 26.81$$

$$= .57\times70 + 26.81$$

$$= 39.90 + 26.81$$

$$= 66.71$$

Now we assume this estimate value of x viz., 66.71 as the given value of x. We estimate the value of y using the regression equation of y on x as follows:

$$y = 1.12x - 5.8$$

$$= 1.12\times66.71 - 5.8$$

$$= 74.7152 - 5.8$$

$$= 68.9152$$

Answer :

The two regression equations are

$$x = .57y + 26.81$$
$$y = 1.12x - 5.8$$
$$x = 66.71 \text{ when } y = 70$$
$$y = 68.9152 \text{ when } x = 66.71$$

8. Regression Coefficients :

Consider the two regression equations x on y and y on x.

$$(x-\bar{x}) = \frac{r\sigma x}{\sigma y}(y-\bar{y}) \qquad (1)$$

$$(y-\bar{y}) = \frac{r\sigma y}{\sigma x}(x-\bar{x}) \qquad (2)$$

Equation (1) is the regression of x on y In this equation consider $\frac{r\sigma x}{\sigma y}$ This is called the coefficient of regression or regression coefficient of x on y That is, coefficient of y* in the regression equation of x on y is called the regression coefficient of x on y.

$(x-\bar{x}) = \frac{r\sigma x}{\sigma y}(y-\bar{y})$ can be written as $1 \times x = \frac{r\sigma x}{\sigma y} \times y - \frac{r\sigma x}{\sigma y} \times \bar{y} + \bar{x}$

Here, 1 is the coefficient of x and $\frac{r\sigma y}{\sigma x}$ is the coefficient of y

Suppose there is a unit change in the value of y. The corresponding change in the value of x is given by the regression coefficient of x on y. That is, when there is a unit change in the value of y, value of x will change by the amount $\frac{r\sigma x}{\sigma y}$

Equation (2) given above is the regression equation of y on x. In this equation consider $\frac{r\sigma y}{\sigma x}$ This is called the coefficient of regression or the regression coefficient of y on x

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* Consider the equation $4x = 2y+3$ In this equation 4 is the number by which the variable x is to be multiplied and it is called the coefficient of x 2 is the number by which the variable y is to be multiplied 2 is called the coefficient of y, in general, in an equation the number by which a variable is to be multiplied is called the coefficient of that variable in the equation

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Check your Progress

4 What are regression coefficients?

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Thus the coefficient of x in the regression of y on x is called the regression coefficient of y on x.

Suppose there is a unit change in the value of x. The corresponding change in the value of y is given by the regression coefficient of y on x. That is, when there is a unit change in the value of x, value of y will change by the amount $\frac{r\sigma y}{\sigma x}$

The formula to calculate the regression coefficient of x on y viz., $\frac{r\sigma x}{\sigma y}$:

$$\frac{r\sigma x}{\sigma y} = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(y-\bar{y})^2}$$

The formula to calculate the regression coefficient of y on x viz., $\frac{r\sigma y}{\sigma x}$ is :

$$\frac{r\sigma y}{\sigma x} = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(x-\bar{x})^2}$$

Example 5:

Using the data given under Example 1 find out the regression coefficient of x on y.

In Example I we found out the values of $\Sigma(x-\bar{x})(y-\bar{y})$ and $\Sigma(y-\bar{y})^2$ as follows:

$\Sigma(x-\bar{x})(y-\bar{y}) = 1568$ $\qquad$ $\Sigma(y-\bar{y})^2 = 2164$

∴ The regression coefficient of x on y is,

$$\frac{r\sigma x}{\sigma y} = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(y-\bar{y})^2}$$

$$= \frac{1568}{2164} = \frac{392}{541}$$

$$= .72$$

Example 6 :

Using the data given under Example 1 find out the regression coefficient of y on x.

In Example I we found out the values of $\Sigma(x-\bar{x})(y-\bar{y})$ and $\Sigma(y-\bar{y})^2$.

$\Sigma(x-\bar{x})(y-\bar{y}) = 1568$

$\Sigma(x-\bar{x})^2 = 1242$

The regression coefficient of y on x is,

$$\frac{r\sigma y}{\sigma x} = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(x-\bar{x})^2}$$

$$= \frac{1568}{1242}$$

$$= \frac{784}{621}$$

$$= 1.26$$

Example 7:

Consider the two regression equations derived in Example 3 and find out two regression coefficients.

The two regression equations derived in Example 3 are

$x = .45y+1$

$y = .8x + 12$

First equation is the regression of x on y. The coefficient of y in the equation is .45, and it is the regression coefficient of x on y. That is,

$$\frac{r\sigma x}{\sigma y} = .45$$

Second equation is the regression of y on x. The coefficient of x in the equation is .8, and it is the regression coefficient of y on x. That is,

$$\frac{r\sigma y}{\sigma x} = .8$$

9. Properties of regression coefficients:

(i) With the help of two regression coefficients we can find out the ratio between σx and σy (i.e) we can find out the value of $\frac{\sigma x}{\sigma y}$

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$$\frac{\sigma x}{\sigma y} = \sqrt{\frac{\text{Regression coefficient of x on y}}{\text{Regression coefficient of y on x}}}$$

$$= \sqrt{\frac{r\sigma x}{\sigma y} - \frac{r\sigma y}{\sigma x}}$$

$$= \sqrt{\frac{r\sigma x}{\sigma y} \times \frac{\sigma x}{r\sigma y}} = \sqrt{\frac{\sigma x^2}{\sigma y^2}} = \frac{\sigma x}{\sigma y}$$

(ii) The two regress on coefficients will have the same sign That is both the regression coefficients will be positive or both will be negative If the regression coefficients are positive correlation will also be positive if the two regression coefficients are negative correlation coefficient will also be negative For σx and σy are always positive

(iii) With the help of the regression coefficients correlation coefficient (r) can be found out.

r^2 = Product of the regression coefficients

$$r = \pm\sqrt{(\text{product of the regression coefficients})}$$

$$= \pm\sqrt{\frac{r\sigma x}{\sigma y} \times \frac{r\sigma y}{\sigma x}}$$

Out of the two signs (plus or minus) put before the square root, plus sign is taken if both the regression coefficients are positive Negative sign is taken if both the regression coefficients are negative

Example 8

The regression equations of x on y and y on x are as follows

$x = 4y+5$

$y = 8x+6$

Find out the correlation coefficient between x and y, the value of $\frac{\sigma x}{\sigma y}$ and the mean values of x and y

The regression equation of x on y is

$x = 4y+5$

Space fc

In this equat on, the coefficient of y viz., 4 is the regression coefficient of x on y,

$$\therefore \frac{r\sigma x}{\sigma y} = 4$$

The regression equation of y on x is

$y = 8x+6$

In this equation the coefficient of x viz 8 is the regression coefficient of y on x

$$\therefore r = \frac{\sigma y}{\sigma x} = 8$$

Here both the regression coefficients are positive

$\therefore$ Correlation coefficient is also positive

$$\therefore r = \sqrt{\frac{r\sigma x}{\sigma y} \times \frac{r\sigma y}{\sigma x}}$$

$$= \sqrt{4 \times 8}$$

$$= \sqrt{32}$$

$$= 5657$$

$$\frac{\sigma x}{\sigma y} = \sqrt{\frac{\text{Regression coefficient of x on y}}{\text{Regression coefficient of y on x}}}$$

$$= \sqrt{\frac{4}{8}}$$

$$= \sqrt{\frac{4}{8}}$$

$$= \sqrt{\frac{1}{2}}$$

$$= \frac{1}{\sqrt{2}}$$

The mean value of x and y are obtained with the help of the following property of regression lines

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The regression lines intersect each other at $(\bar{x}, \bar{y})$, that is, if we find out the values of x and y from the regression equations, these values will be $\bar{x}$ and $\bar{y}$ respectively.

Hence, to find out the values of $\bar{x}$ and $\bar{y}$, it is enough if we find the values of x and y from the given regression equations.

Values of x and y are obtained from the given regression equations as follows:

$$x = .4y+5$$

$$y = .8x+6$$

Let us put the value of x viz., (.4y+5) in the second equation viz.,

$$\begin{aligned}
y &= .8x+6 \text{ as follows:} \\
y &= .8(.4y+5)+6 \\
&= (.8\times.4y)+(.8\times5)+6 \\
&= .32y+4+6 \\
&= .32y+10 \\
y-.32y &= 10 \\
.68y &= 10 \\
\therefore y &= \frac{10}{.68} = \frac{1000}{68} \\
&= 14.7 \\
\therefore x &= .4y+5 \\
&= (.4\times14.7)+5 \\
&= 5.88+5 \\
&= 10.88
\end{aligned}$$

The values of x and y are the values of $\bar{x}$ and $\bar{y}$ respectively

$$\therefore \bar{x} = 10.88$$

$$\bar{y} = 14.7$$

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Example 9:

The regression lines of y on x and x on y are given below

$y = .80x+25$

$x = .45y+30$

Find the correlation coefficient between x and y and the arithmetic means of x and y. Also show that $\sigma x : \sigma y = 3:4$

The regression equation of y on x is

$$y = .80x+25$$

$$\therefore\ r\frac{\sigma y}{\sigma x} = \text{regression coefficient of y on x.}$$

$$= \text{coefficient of x in the regression equation of y on x.}$$

$$= .8$$

The regression equation of x on y is

$$x = .45y+30$$

$$\therefore\ \frac{r\sigma x}{\sigma y} = \text{regression coefficient of x on y.}$$

$$= \text{coefficient of y in the regression equation of x on y.}$$

$$= .45$$

Both the regression coefficients are positive.

$\therefore$ Correlation coefficient is also positive.

$$\therefore r = +\sqrt{\frac{r\sigma x}{\sigma y} \times \frac{r\sigma y}{\sigma x}}$$

$$= \sqrt{.45 \times .8}$$

$$= \sqrt{.360}$$

$$= .6$$

Space for hints

$$\frac{\sigma x}{\sigma y} = \sqrt{\frac{\text{Regression coefficient of } x \text{ on } y}{\text{Regression coefficient of } y \text{ on } x}}$$

$$= \sqrt{\frac{.45}{.8}}$$

$$= \sqrt{\frac{45}{80}}$$

$$= \sqrt{\frac{9}{16}} = \frac{3}{4}$$

$$\therefore \sigma_X : \sigma_Y = 3 : 4$$

Let us put the value of y viz., (.8x+25) in the second equation viz.,

$$x = .45y + 30$$

$$\therefore x = .45(.8x+25) + 30$$

$$= (.45\times.8x) + (.45\times25) + 30$$

$$= .360x + 11.25 + 30$$

$$= .360x + 41.25$$

$$x - .36x = 41.25$$

$$.64x = 41.25$$

$$\therefore x = \frac{41.25}{.64} = \frac{4125}{64} = 64.45$$

$$\therefore y = .8x + 25$$

$$= .8\times64.45 + 25$$

$$= 51.560 + 25$$

$$= 76.56$$

These values of x and y are the values of $\bar{x}$ and $\bar{y}$ respectively

$$\therefore \bar{x} = 64.45$$

$$\bar{y} = 76.56$$

Example 10 :

The following are the two regression equations:

$x+2y = 5$

$2x+3y = 8$

Find out the correlation coefficient and the value of the ratio $\sigma x{:}\sigma y$. Also find out the mean values.

To find out the correlation coefficient we should know the two regression coefficients. This we can get from regression equations given to us. Since we do not know as to which of the two equations is the regression equation of x on y, we assume the first equation given to be the regression equation of x on y.

So let us rewrite the two equations as follows:

$$x = -2y+5$$

$$y = -\frac{2}{3}x+\frac{8}{3}$$

Coefficient of y in the first equation is the regression coefficient of x on y.

$$\therefore \frac{r\sigma x}{\sigma y} = -2$$

Coefficient of x in the second equation is the regression coefficient of y on x.

$$\therefore \frac{r\sigma y}{\sigma x} = -\frac{2}{3}$$

Now,

$$r^2 = \text{Product of the two regression coefficients}$$

$$= (-2) \times \left(-\frac{2}{3}\right)$$

$$= \frac{4}{3}$$

$$= 1.33 \text{ which is greater than one.}$$

But value of r^2, is always less than one. Therefore, our assumption viz, the first equation is the regression equation of x on y and the second

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equation is the regression equation of y on x is wrong. Only the second equation is the regression equation of x on y.

Now, we rewrite the two equations as follows to get the two regression coefficients:

$$y = \left(-\frac{1}{2}\right)x + \frac{5}{2}$$

$$x = \left(-\frac{3}{2}\right)y + 4$$

$$\therefore \frac{r\sigma y}{\sigma x} = \left(-\frac{1}{2}\right)$$

$$r\frac{\sigma y}{\sigma x} = \left(-\frac{3}{2}\right)$$

$$\therefore r^2 = \left(-\frac{3}{2}\right) \times \left(-\frac{1}{2}\right)$$

$= 3/4$ which is less than one.

Since the two regression coefficients are negative r is also negative.

$$\therefore r = -\sqrt{\frac{r\sigma x}{\sigma y} \times \frac{r\sigma y}{\sigma x}}$$

$$= -\sqrt{\left(\frac{3}{2}\right) \times \left(\frac{1}{2}\right)}$$

$$= -\frac{\sqrt{3}}{2}$$

$$= -\frac{1.732}{2} = -.866$$

$$\frac{\sigma x}{\sigma y} = \sqrt{\frac{\text{Regression coefficient of x on y}}{\text{Regression coefficient of y on x}}}$$

$$= \sqrt{\left(-\frac{3}{2}\right) \div \left(-\frac{1}{2}\right)}$$

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$$= \sqrt{\frac{\frac{3}{2}}{\frac{1}{2}}}$$

$$= \sqrt{3}$$

$$\sigma x : \sigma y = \sqrt{3} : \sqrt{1} = \sqrt{3} : 1$$

To get the mean values, we get the values of x and y from the two regression equations given as follows:

Let us put the value of $y = \left(-\frac{1}{2}\right)x + \frac{5}{2}$ in the second equation.

viz., $x = \left(-\frac{3}{2}\right)y + 4$ as follows:

$$x = -\frac{3}{2}\left(-\frac{1}{2}x+\frac{5}{2}\right) + 4$$

$$= \left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)x + \left(-\frac{3}{2}\right) + \left[\frac{5}{2}\right] + 4$$

$$= \frac{3}{4}x - \frac{15}{4} + 4$$

$$= \frac{3x-15+16}{4}$$

$$= \frac{3x+1}{4}$$

$$\therefore 4x = 3x+1$$

$$4x - 3x = 1$$

$$x = 1$$

$$\therefore y = -\frac{1}{2}x + \frac{5}{2}$$

$$= -\frac{1}{2}\times 1 + \frac{5}{2}$$

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$$= -\frac{1}{2}+\frac{5}{2}$$

$$= \frac{-1+5}{2}$$

$$= \frac{4}{2}$$

$$= 2$$

$$\therefore \ \bar{x} = 1, \quad \bar{y} = 2$$

Example 11:

The regression lines of y and x are given below:

$$y = .9x+2.3$$

$$x = .4y+.86$$

Find the means of x and y and coefficient of correlation between them.

Regression equation of y on x is

$$y = .9x+2.3$$

$\therefore$ Regression coefficient of y on x.

$$\frac{r\sigma y}{\sigma x} = .9$$

Regression equation of x on y is,

$$x = .4y+ .86$$

$\therefore$ Regression coefficient of x on y

$$\frac{r\sigma x}{\sigma y} = .4$$

$$\therefore \ r = \sqrt{\frac{r\sigma y}{\sigma x}\times\frac{r\sigma x}{\sigma y}}$$

$$= \sqrt{.9\times.4}$$

$$= \sqrt{.36}$$

$$= .6$$

$\therefore$ Correlation coefficient $= .6$

Putting the value of y viz., (.9x+2.3) in the second equation we get

$$x = .4(.9x+2.3)+.86$$

$$= .36x+.92+.86$$

$$x-.36x = 1.78$$

$$\text{(i.e.,) } .64x = 1.78$$

$$\therefore\ x = \frac{1.78}{.64}$$

$$= \frac{178}{64}$$

$$= \frac{89}{32}$$

$$= 2.78$$

$$\therefore\ y = .9x+2.3$$

$$= .9\times 2.78+2.3$$

$$= 2.502+2.3$$

$$= 4.802$$

$$\therefore\ \bar{x} = 2.78$$

$$\bar{y} = 4.802$$

10. Utility or importance of Regression Analysis:

1. One of the principal uses of regression analysis is prediction and because of this use greater emphasis has been shifted from correlation to regression in rècent years. Regression analysis has growing importance in Economics.

2. Regression is used to estimate supply and demand according to change in price. It is useful in estimating the likely increase in consumption and savings with increase in income.

3. Estimation of the production values of farms on the basis of their characteristics are made by regression analysis in order to levy taxes and grant loans.

Space for hints

Space for hints

4. It has wide application in the study of savings and investment in the field Econometrics.

5. The influence of varying weather conditions on the yield of crops is analysed by regression methods.

6. The influence of physical inputs on the output of various crops are studied by regression analysis.

7. Regression analysis has been used in the study of relationship between technological progress and economic growth.

8. Estimates of the change in public income due to changes in rates of taxation and estimates of the change in bank deposits and bank loans due to the increase in the bank rates of interest are all made by regression analysis.

9. The influence of foreign exchange rates on the amount of imports and exports is studied by regression analysis.

11. Difference between Correlation and Regression:

Though correlation and regression are closely related to each other, there are essential differences between them.

1. Correlation gives us the nature and degree of relationship between two variables x and y. But, regression gives us the average change in the value of one variable when there is a change in the value of the other variable. That is, regression gives us the exact relationship between two variables.

Check your Progress

5. What is the main difference between correlation and regression?

2. If x and y are the two variables given, correlation between x and y is the same as the correlation between y and x. Thus correlation is a two way relationship. But, in the case of regression, regression of x on y is not the same as the regression of y on x. Thus regression is a one way relationship.

12. Answers to Check Your Progress Questions :

1. Refer 1
2. Refer 3
3. Refer 4
4. Refer 8
5. Refer 11

13. Model questions for guidance :

Space for hints

10 Marks Questions (One Page Answer)

1. Explain the concept of correlation and regression and bring out the relationship between them.

2. What are regression coefficients and how do they differ from the correlation coefficient?

3. Given the following values:

	Mean	S.D
Yield of wheat (kg/unit area)	10	8
Annual rainfall (inches)	8	2

Correlation coefficient 0.5

Estimate the yield when rainfall is 9 inches.

4. Distinguish between correlation and regression.

20 Marks Questions (Three Page Answer)

1. The regression equations are

$x = -.2y+4.2$

$y = -.8x+8.4$

Calculate (1) $\bar{x}$ and $\bar{y}$ (ii) r (iii) $\sigma x: \sigma y$ (iv) the proable value of y when x = 4

2. The following are two regression equations

$3x+2y = 26$

$6x+ y = 31$

Find the mean values, correlation coefficient between x and y and the ratio between the s.d of y and x.

3. Obtain the regression equations of income on years of service.

Years of Service :	11	7	9	5	3	6	8
Income:	7	5	3	2	4	4	3

Space for hints

4. Find the co-efficient of correlation between x and y from the following data.

x	3	6	5	4	3	6	7	5
y	3	2	3	5	3	6	6	4

Also calculate the regression of y on x and predict the average value of y when x is 9.

5.

	Calcutta	Bombay
Mean Price	Rs.65	Rs.67
Standard Deviation	Rs.2.50	Rs.3.50

Coefficient of Correlation between the prices of Calcutta and Bombay = 0.8 From the above data,

a) Write down the two regression lines.

b) Obtain the best estimate of price in Bombay when the price in Calcutta is Rs.70.

6. Estimate the two regression lines and slso the correlation coefficient from the following data:

$\Sigma x = 30$ $\quad \Sigma y = 40$ $\quad \Sigma xy = 214$

$\Sigma x^2 = 220$ $\quad \Sigma y^2 = 340$ $\quad N = 5$

UNIT - 6

INDEX NUMBERS

Space for hints

Introduction

When we have a single variable or group of variables together changing from time to time or from place to place and when we wish to study the relative change of this variable or this group of variables, we use the device known as index numbers. Changes taking place in the economic variables like general price level prevailing in a country, volume of foreign trade, standard of living of the population and capital formation cannot be measured directly, but can be measured indirectly with the help of the statistical tool, namely index numbers. What are index numbers? How are they computed? Answers to such questions are explained in this Unit-6.

Unit Objectives :

After studying this Unit, you would be able to understand

* the meaning of index numbers
* the various types of index numbers and their computation
* the steps involved in the construction of wholesale price index numbers
* the uses and limitations of index numbers.

Unit Structure :

1. Index Numbers - Meaning and Definition
2. Averages versus Index Numbers
3. Evolution of Index Numbers
4. Construction of Price Index Number for single commodity
5. Constructionof Quantity Index Number for single commodity
6. Constructionof Index Number for a group of commodities
7. Unweighted Index Numbers
8. Weighted Index numbers :
9. Tests of Consistency for an Index Number
10. Construction of Wholesale Price Index Number
11. Usefulness of Index Numbers
12. Limitations of Index Numbers
13. Answers to Check Your Progress Questions
14. Model questions for guidance.

1. Index Numbers - Meaning and Definition

Suppose we have a hundred commodities whose prices are known from time to time over a period of years, one may change more and another less. But the group of this hundred together may have a slow movement. This group movement will not be generally revealed by studying the individual changes separately. The Index number is a useful concept for the study of this group movement.

Space for hints

An index number may be described as a specialized average designed to measure the relative change in the level of a phenomenon from time to time or from place to place.

2. Averages versus Index Numbers

An average is a single figure representing a group of figures; this group of figures must be comparable and also be in the same unit of measurement. Average height of men, average height of women and average height of children in a particular locality are meaningful. On the other hand, average height of all men, women and children taken together is meaningless because men, women and children form three different categories of people and heights of a man and a child or of a woman and a child or of a man and a woman are not comparable.

Again when we consider the average height of some men given in inches and of others given in cms., the average calculated for all men taken as a simple group will be meaningless. So an average of a group figures is meaningful only when the figures are comparable and are in the same unit of measurement.

On the other hand, index number is used to get an average of different types of items which may be expressed in different units of measurement also. Let us suppose that we have to study the group movement when the prices of three commodities over a period of time are given. Let rice, cloth and coal be the three given commodities. These are three entirely different commodities. Also the price of rice may be given in **Rs. per measure** where as the price of cloth be given in **Rs. per metre** and of coal in **Rs. per ton**. Thus, the units are also different. Even then the technique of index number gives an average price of these different types of commodities expressed in different units. Hence index number is considered to be a specialized average.

3. Evolution of Index Numbers

The technique of index numbers, historically, was invented in the eighteenth century by persons like Dutot and Carli. They constructed index numbers for measuring changes in prices. In 1865, W.S. Jevons made use of index numbers to measure the changes in the price levels and thereby, the value of money. Now it has assumed greater significance as with other statistical methods like averages, measures of dispersion etc. It is now used to measure changes in any economic phenomenon. Hence we have different kinds of index numbers. Important types of index numbers

(i) Price index numbers

(ii) Quantity index numbers

(iii) Value index numbers

(iv) Special purpose index numbers

Check your Progress

1. What is an index number?

Space for hints

One who is familiar with the construction of price and quantity index numbers can easily construct value and special purpose index numbers. So, we discuss in detail about price and quantity index numbers only.

4. Construction of Price Index Number for single commodity

Suppose we are given that the price of rice in 1980 is Rs.2 per measure and in 1982 it is Rs.3 per measure. Suppose we want to study the price movement of rice. For this, we calculate the index numbers as follows :

We equate the price of rice in 1980 to 100. We calculate the price of rice in 1982 as a percentage of the price in 1980. This percentage value is the index number of price (or price index) for the year 1982.

$$\therefore \text{ Price index for 1982} = \frac{\text{Price of rice in 1982}}{\text{Price of rice in 1980}} \times 100$$

$$= \frac{3}{2} \times 100 = 150$$

4.1 Base year and Base year price

The year for which we have equated the price to 100 is called the *'base year'* and the price in that year is called the 'base year's price'. So, in our Example, 1980 is the base year and Rs.2 is the base year's price.

4.2 Current year and Current year price

The year for which we calculate the index number is called the 'current year' and the price in that year is called the 'current year's price'. In our Example, 1982 is the current year and Rs.3 is the current year's price.

4.3 The formula to calculate the price index

$$\boxed{\text{Price Index} = \frac{\text{Current year's price}}{\text{Base year's price}} \times 100}$$

The index number for the base year is always equal to 100.

4.4 Procedure to measure change in price

To measure the change in the price in a particular year, the difference between the index number of the given year and that of the base year (viz, 100) is to be found out. In our example, the change in the price of rice in 1982 = 150 – 100 = 50, We say that there is 50% increase in the price of rice in 1982 from its price in 1980.

4.5 Price Relative

The ratio, $\frac{\text{Current year's price}}{\text{Base year's price}}$ is called price relative.

Space for hints

4.6 Percentage Price Relative

Price relative multiplied by 100 is called percentage price relative. So, it is to be noted here that the price index is equal to the percentage price relative.

Symbolic Representation of Price Index

Usually price is represented by the letter P. As we have already noted, we have two years viz, base year and current year when we calculate the index number. Now the subscript 0 is always used to represent base year and subscript1 is used to represent current year. Therefore the base year's price is denoted by the symbol P_0, and the current year's price is denoted by the symbol P_1.

Price index is usually denoted by the symbol P_{01}. Therefore the formula to get the price index in the case of a single commodity is as follows.

$$\boxed{P_{01} = \frac{P_1}{P_0} \times 100}$$

Example-1 :

Taking 1965 as base year calculate the price index numbers for the years 1966, 1967 and 1968 from the following data.

Year	1965	1966	1967	1968
Price of a steel chair in Rupees	25	27	30	35

We are asked to take 1965 as the base year.

$$\therefore\ P_0 = 25$$

When we calculate the index number for 1966, 1966 is the current year and $P_1 = 27$.

$$\therefore \text{ For 1966, price index} = \frac{P_1}{P_0} \times 100 = \frac{27}{25} \times 100$$

$$= 108$$

Similarly, when we calculate index number for 1967, 1967 is the current year and hence $P_1 = 30$.

$$\therefore \text{ Price index for 1967} = \frac{P_1}{P_0} \times 100 = \frac{30}{25} \times 100$$

$$= 120$$

When we calculate index number for 1968, 1968 is the current year and $P_1 = 35$.

$$\therefore \text{ Price index for 1968} = \frac{P_1}{P_0} \times 100 = \frac{35}{25} \times 100$$

$$= 140$$

Check your Progress

2. What is price relative?

3. How do you compute percentage price relative?

Now, we can give the above index numbers in tabular form as follows :

Space for hints

Year	Price of steel chair in Rupees	Price Index
1965	25	100
1966	27	108
1967	30	120
1968	35	140

5. Constructionof Quantity Index Number for single commodity

Quantity index numbers are constructed in exactly the same manner as the price index numbers are constructed.

Suppose the quantity of coal produced in 1957 is 4 billion tons and in 1958, it is 7 billion tons. To study the movement of the quantity of coal produced, we calculate quantity index numbers as follows :

Just as in the case of price index numbers, we equate the quantity produced in 1957 to 100 and call the year 1957 as base year. We call the quantity figure for 1957 viz, 4 as the base year's quantity.

Now we give the formula to get the quantity index as follows

$$\text{Quantity index number} = \frac{\text{Current year's quantity}}{\text{Base year's quantity}} \times 100$$

$$\therefore \text{ Quantity index for 1958} = \frac{7}{4} \times 100 = 175$$

The ratio $\frac{\text{Current year's quantity}}{\text{Base year's quantity}}$ is called the quantity relative and quantity relative multiplied by 100 is called the percentage quantity relative.

The change in quantity produced is also measured in exactly the same manner as we have measured the change in price level.

The difference between the quantity index of a particular year and the base year index (viz, 100) gives the change in the quantity in the given year from the base year.

$\therefore$ Change in the quantity produced in 1958 = 175 – 100 = 75%

5.1 Symbolic Representation of Quantity Index

To represent quantity, the letter Q is always used. As we have already noted subscript 0 is used to represent the base year and subscript 1 is used to represent the current year. Therefore Q_0 represents base year's quantity and Q_1 represents current year's quantity.

Space for hints

Quantity index is usually denoted by Q_{01}. Therefore, the formula to get the quantity index in the case of a single commodity is as follows :

$$Q_{01} = \frac{Q_1}{Q_0} \times 100$$

It is to be noted that we get the quantity index if we put the word quantity wherever we have the word price in the case of price index. That is, if we replace P by Q (subscripts remaining constant) in the price index we get the formula to calculate the quantity index.

Example-2 :

Compute the Quantity Index for 1958.

	Crude oil produced in Billion Barrels
1957	5
1958	4

1957 is the base year.

$\therefore\ Q_0 = 5.$

1958 is the current year.

$\therefore\ Q_1 = 4.$

$$\text{Quantity index number} = \frac{Q_1}{Q_0} \times 100$$

$$= \frac{4}{5} \times 100$$

$$= 80.$$

6. Constructionof Index Number for a group of commodities

So far we have considered the construction of index numbers in the case of a single commodity. But often we come across a group of commodities and we wish to study the change in the price (or quantity) of the group as a whole. In such cases the index numbers are not simply the percentage price (or quantity) relatives. To calculate the index numbers for the group as a whole, several people have given several formulae. These formulae can broadly be classified into two groups as follows:

(i) Unweighted index numbers

(ii) Weighted index numbers.

7. Unweighted Index Numbers

If we give equal importance to each and every commodity given and calculate the index numbers, they are called unweighted index numbers.

The unweighted index numbers, can further be divided into two groups as follows :

Space for hints

(A) Simple aggregative

(B) Simple average of relatives

7.1 Simple aggregative method :

Step by step procedure for the calculation of price index by simple aggregate method.

(i) Fix the base year first. If there is any specification regarding the base year, fix it accordingly. Otherwise, choose the first of the given years as the base year.

(ii) Denote the prices guven for the base year P_0.

(iii) Add the prices of the given commodities for the base year. This sum is denoted by SP_0.

(iv) The year for which we are going to calculate the index is the current year.

(v) Denote the current year prices by P_1.

(vi) Add the prices of the same commodities for the current year. It is denoted by ΣP_1.

(vii) Now the prices index is given by the formula,

$$P_{01} = \frac{\Sigma P_1}{\Sigma P_0} \text{ X } 100$$

P_{01} gives the required index number.

Example-3 :

Calculate the simple aggregate price index for the following group of commodities taking 1965 as the base year.

Commodity and unit	Price in 1965 (Rs.)	Price in 1970 (Rs.)
Butter per Kg.	10.00	12.00
Milk per Litre	1.20	1.50
Ghee per Tin	19.00	19.80
Bread per Kg.	1.40	1.80
Eggs per Dozen	3.00	3.50

We are asked to take 1965 as the base year. So the sum of the prices for 1965 is the value of ΣP_0.

We have to get the index number for 1970. Therefore, 1970 is the current year

Check your Progress

4. Give the formula to calculate simple aggregative price index.

Space for hints

and sum of the prices for 1970 is the value of ΣP_1. We get the values of ΣP_0 and ΣP_1 as follows :

Commodity	Price in 1965 P_0	Price in 19701 P_1
Butter per Kg.	10.00	12.00
Milk per Litre	1.20	1.50
Ghee per Tin	19.00	19.80
Bread per Kg.	1.40	1.80
Eggs per Dozen	3.00	3.50
Total	$\Sigma P_0 = 34.60$	$\Sigma P_1 = 38.60$

$$\therefore\ P_{01} = \frac{\Sigma P_1}{\Sigma P_0} \times 100$$

$$= \frac{38.60}{34.60} \times 100$$

$$= \frac{193}{173} \times 100 = 111.6$$

$\therefore$ Simple aggregate price index for 1970 = 111.6

Here we can state that the price of the group of commodities given in 1970 is 111.6% and the increases in the price of the group is 11.6%.

Disadvantages of simple aggregate method :

The chief disadvantage of this method is that some items get importance because they are quoted in a particular unit. For instance, ghee is quoted per tin in the above example. Instead of that if the price is expressed per kg. we may get the index to be very different from the one we have got earlier. Thus, the units in which the price of commodities are quoted affect the price index. Further equal importance is given to all the items irrespective of their relative importance. Therefore, this index is not an objective measure of change in prices.

7.2 Simple average of relatives method :

Step by step procedure

(i) When we are given the prices of certain commodities in two years, we find out the percentage price relative viz., $\frac{P_1}{P_0} \times 100$ for each commodity.

(ii) Then, we sum up these values and denote the sum by $\Sigma \frac{P_1}{P_0} \times 100$

(iii) The number of commodities given is denoted by n.

Space for hints

(iv) Now, the simple average of price relatives is computed using either arithmetic mean or geometric mean.

(a) Simple average price index using arithmetic mean :

$$P_{01} = \frac{\Sigma \frac{P_1}{P_0} \times 100}{n}$$

(b) Simple average price index using geometric mean is used,

$$P_{01} = \text{Anti-log} \frac{\left\{\Sigma \log \left(\frac{P_1}{P_0} \times 100 \right) \right\}}{n}$$

Let us illustrate the procedure to find out the index number by the method of simple average of relatives by the following example.

Example-4 :

Prices of six commodities A, B, C, D, E, F, in two years in 1968 and 1969 are as follows :

Commodity	Price per Kg. in Rs.	
	1968	1969
A	7.30	7.70
B	7.70	5.50
C	7.00	8.00
D	6.50	7.30
E	34.10	29.80
F	17.30	17.10

First let us find out the index number by averaging the price relatives by arithmetic mean.

Let us take the year 1968 as the base year and 1969 as the current year. Now for each commodity the price relative is calculated as shown on the Table below:

Commodity	P_0	P_1	$\frac{P_1}{P_0} \times 100$
A	7.30	7.70	105.5
B	7.70	5.50	71.4
C	7.00	8.00	114.3
D	6.50	7.30	112.3
E	34.10	29.80	87.4
F	17.30	17.10	99.0
		Total	589.9

Check your Progress

5. How do you calculate simple average price index?

Space for hints

Number of commodities given $= n$

$= 6$

$$\Sigma \frac{P_1}{P_0} \times 100 = 589.9$$

Simple average price number $= \dfrac{\Sigma \frac{P_1}{P_0} \times 100}{n}$

$= \dfrac{589.9}{6}$

$= 98.3$

We find out the index number by averaging the price relatives by the geometric mean as follows:

Commodity	P_0	P_1	$\frac{P_1}{P_0} \times 100$	$\log\left[\frac{P_1}{P_0} \times 100\right]$
A	7.30	7.70	105.5	2.0232
B	7.70	5.50	71.4	1.8537
C	7.00	8.00	114.3	2.0581
D	6.50	7.30	112.3	2.0504
E	34.10	29.80	87.4	1.9415
F	17.30	17.10	99.0	1.9956
			Total	11.9225

$$\therefore \Sigma \log\left[\frac{P_1}{P_0} \times 100\right] = 11.9225$$

n = number of commodities given

$= 6$

$$\frac{\Sigma \log\left[\frac{P_1}{P_0} \times 100\right]}{n} = \frac{11.9225}{6}$$

$= 1.9871$ (approx.)

$$\text{Now, } P_{01} = \text{Anti-log} \frac{\left\{\Sigma \log\left(\frac{P_1}{P_0} \times 100\right)\right\}}{n}$$

$=$ Anti-log 1.9871

$= 97.07$

It is to be noted that the two index numbers are slightly different. When we use arithmetic mean, the index is 98.3 and when we use geometric mean the index is 97.07. This is due to the fact that arithmetic mean will always be greater than geometric mean.

Space for hints

7.3 G.M. is preferred to A.M. in the calculation of Index Numbers:

In the problems of index numbers, we are interested in the relative changes rather than in the absolute changes in the values of variables. Geometric mean measures relative changes whereas arithmetic mean measures only absolute changes. Therefore, the index got by the geometric mean would be more accurate compared to the index got by arithmetic mean. Hence, in the calculation of index numbers geometric mean is preferred to arithmetic mean. But arithmetic mean is widely used for its simplicity of calculation.

Example-5 :

Find out the index number of price of the group of commodities given for 1980 taking 1970 as the base year.

(i) by the simple aggregate of actual prices method.

(ii) by the simple average of price relatives method using arithmetic mean as well as geometric mean.

Commodity and units	Price in 1970 Rs.	Price in 1980 Rs.
Rice per measure	1.25	2.75
Gram per measure	0.75	1.75
Sugar per kilo	1.25	1.75
Oil per litre	2.25	6.25
Vegetable per kilo	0.50	0.70

(i) Method of Simple aggregate of actual prices :

1970 is the base year and 1980 is the current year.

$\therefore$ $S P_0$ = Sum of the base year's prices

= Sum of the prices in 1970

= Rs. [1.25 + .75 + 1.25 + 2.25 + 0.50]

= Rs.6.00

ΣP_1 = Sum of the current year's prices

= Sum of the prices in 1980

= Rs. (2.75 + 1.75 + 1.75 + 6.25 + 0.70)

= Rs. 13.20

$$\text{Index number} = \frac{\Sigma P_1}{\Sigma P_0} \times 100$$

Space for hints

$$= \frac{13.20}{6} \times 100$$

$$= 220$$

(ii) (a) (i) Method of averaging the price relatives by the arithmetic mean :

Since 1970 is the base year, the prices in 1970 are given under the heading P_0 in the table below. Since 1980 is the current year the prices in 1980 are given under the heading P_1 in the table below. For each commodity we have found out the percentage price relative and given under the heading $\frac{P_1}{P_0} \times 100$ in the table below.

Item	P_0	P_1	$\frac{P_1}{P_0} \times 100$
Rice	1.25	2.75	$\frac{2.75}{1.25} \times 100 = 220.0$
Gram	.75	1.75	$\frac{1.75}{.75} \times 100 = 233.3$
Sugar	1.25	1.75	$\frac{1.75}{1.25} \times 100 = 140.0$
Oil	2.25	6.25	$\frac{6.25}{2.25} \times 100 = 277.7$
Vegetable	.50	.70	$\frac{.70}{.50} \times 100 = 140.0$
			Total = 1011.0

$$\therefore \Sigma \frac{P_1}{P_0} \times 100 = 1011$$

n = Total number of commodities given

= 5.

$$\therefore \text{Index number} = \frac{\Sigma \frac{P_1}{P_0} \times 100}{n}$$

$$= \frac{1011}{5}$$

$$= 202.2$$

(b) Method of averaging the price relatives by the geometric mean :

Here, we find out the logarithm of each value of $\frac{P_1}{P_0} \times 100$

and give under the heading $\log \left[\frac{P_1}{P_0} \times 100 \right]$ in the following table .

Item	$\frac{P_1}{P_0} \times 100$	$\log\left(\frac{P_1}{P_0} \times 100\right)$
Rice	220.0	2.3424
Gram	233.3	2.3680
Sugar	140.0	2.1461
Oil	277.7	2.4436
Vegetable	140.0	2.1461
		Total = 11.4462

$$\therefore \Sigma \log\left(\frac{P_1}{P_0} \times 100\right) = 11.4462$$

$$n = \text{number of commodities given}$$

$$= 5$$

$$\therefore \text{Index number} = \text{Anti-log} \frac{\left\{\Sigma \log\left(\frac{P_1}{P_0} \times 100\right)\right\}}{n}$$

$$= \text{Anti-log}\left(\frac{11.4462}{5}\right)$$

$$= \text{Anti-log}\ (2.2892)$$

$$= 194.6$$

$$\therefore \quad P_{01} = 194.6$$

7.4 Calculation of Quantity Index Numbers

Quantity index numbers can also be constructed using both the methods given above and formulae are as follows:

7.4.1 Simple aggregate method :

$$Q_{01} = \frac{\Sigma Q_1}{\Sigma Q_0} \times 100$$

7.4.2 Simple average of quantity relatives method :

(a) Simple index by arithmetic mean :

$$Q_{01} = \frac{\Sigma \frac{Q_1}{Q_0} \times 100}{n}$$

(b) Simple average index by geometric mean :

$$\therefore \text{Index number} = \text{Anti-log} \frac{\left\{\Sigma \log\left(\frac{Q_1}{Q_0} \times 100\right)\right\}}{n}$$

Space for hints

Space for hints

Just as P_1 and P_0 denoted current year and base year prices, here Q_1 and Q_0 denote current year and base year quantities respectively. Here also n denotes the number of commodities given.

These formulae are obtained by replacing P_1 and P_0 in the price index by Q_1 and Q_0 respectively.

Example-7 :

Quantity produced in two years viz, 1957 and 1958 of two fuels viz, coal and crude oil are given below. Compute the quantity index for 1958 by all the three methods given above :

Item and Unit	Quantity Produced (Billion)	
	1957	1958
Coal (ton)	3	2
Crude oil (Barrel)	4	4

1957 is the base year and hence the quantities in 1957 are given under Q_0 in the table below.

1958 is the current year and hence the quantities in 1958 are given under Q_1 in the table below.

ΣQ_0 = Sum of quantities in the base year

= Sum of quantities in 1957

$= 3 + 4$

$= 7.$

ΣQ_1 = Sum of quantities in the current year

= Sum of quantities in 1958.

$= 2 + 4$

$= 6.$

By the simple aggregate method,

$$\text{Quantity index} = \frac{\Sigma Q_1}{\Sigma Q_0} \times 100$$

$$= \frac{6}{7} \times 100$$

$$= 85.7$$

To calculate by the method of averaging by arithmetic mean.

Item	Q_0	Q_1	$\frac{Q_1}{Q_0} \times 100$
Coal	3	2	66.7
Crude oil	4	4	100.0
Total			166.7

$$\therefore \quad \Sigma\frac{Q_1}{Q_0} \times 100 = 166.7$$

$$n = \text{number of commodities} = 2$$

$$\therefore \text{Index number} = \frac{\Sigma \frac{Q_1}{Q_0} \times 100}{n} = \frac{166.7}{2} = 83.35$$

To calculate the index number by the method of averaging by geometric mean.

Item	$\frac{Q_1}{Q_0} \times 100$	$\log\left(\frac{Q_1}{Q_0} \times 100\right)$
Coal	66.7	1.8241
Crude oil	100	2.0000
	Total	3.8241

$$\therefore \Sigma \log\left(\frac{Q_1}{Q_0} \times 100\right) = 3.8241$$

$$\frac{\Sigma \log\left(\frac{Q_1}{Q_0} \times 100\right)}{n} = \frac{3.8241}{2} = 1.9121 \text{ (approx.)}$$

$$\therefore \text{Index number} = \text{Anti-log} \frac{\left\{\Sigma \log\left(\frac{Q_1}{Q_0} \times 100\right)\right\}}{n} = \text{Anti-log } (1.9121) = 81.68$$

Space for hints

Space for hints

7.5 Average method versus Aggregate method

The index of simple average of relatives method is superior to the index of simple aggregate method in the following respect:

The influence due to different units in which the commodities are given is completely removed.

But both the methods have the drawback viz., that they do not assign weights to different items according to their relative importance.

8. Weighted index numbers :

Index numbers calculated by assigning weights (according to the relative importance) to various items given are called weighted index numbers.

8.1 Need for weights :

Suppose the price of rice has doubled while the price of cashewnuts has increased tenfold. Here, simple index number of these two prices viz., $\frac{200+1000}{2} = \frac{1200}{2} = 600$ indicates that the general level of price of these two articles to a middle class man is very different. It is necessary that a middle class man has to spend double the amount to buy rice which is his principal diet. But the importance of cashewnuts for him as food is so little that it does not affect him at all. Hence, proper weights are to be given to each of the commodities according to the purpose of the index number so that the weighted index will serve the purpose much better.

8.2 Methods of weighting

The weights to the items can be assigned by whatever seems appropriate to bring out their economic importance. For example, weights can be quantities produced or consumed or distribution figures in the case of price index. There are three methods of weighting index numbers.

(i) Price weights : In this case, the items included in the index number are given importance according to their price.

(ii) Quantity weights : In this case, the items included in the index number are given importance according to the amount of quantity used, consumed : purchased, sold etc.

(iii) Value weights : In this case all the items are given importance according to the expenditure involved on them.

The weights can be taken from the base period or from the current period or

Check your Progress

6. What do you understand by weighted index number?

7. Name the different types of weights used.

can be a combination of quantities from both periods. For each type of weighting we get an index number. We discuss about only a few important and commonly used weighted index numbers.

Space for hints

8.3 Laspeyre's Index number :

8.3.1 Definition

The weighted aggregate price index with base year quantity as weight or quantity index with base year price as weight is called the Laspeyre's index. The formulae are:

$$P_{01} = \frac{\Sigma Q_0 P_1}{\Sigma Q_0 P_0} \times 100$$

$$Q_{01} = \frac{\Sigma P_0 Q_1}{\Sigma P_0 Q_0} \times 100$$

So, Laspeyre's price index can be obtained if we are given the quantity figures for the base year and the price figures for both the current year and the base year.

Laspeyre's quantity index can be obtained if we are given the price figures for the base year and the quantity figures for both the current year and base year.

8.3.2 Alternative expression

Laspeyre's formula can be rewritten as weighted average of relatives as follows.

$$P_{01} = \frac{\Sigma Q_0 P_0 \times \frac{P_1}{P_0}}{\Sigma Q_0 P_0} \times 100$$

$$Q_{01} = \frac{\Sigma P_0 Q_0 \times \frac{Q_1}{Q_0}}{\Sigma P_0 Q_0} \times 100$$

Hence, using these formulae, Laspeyre's price index is obtained by calculating the weighted arithmetic mean of the price relatives taking base year's values* as weights.

Laspeyre's quantity index is obtained by calculating the weighted arithmetic mean of the quantity relatives taking base year's value as weights.

So, Laspeyre's price index formulae given in the second form is used when we are given the base year's value figures and the price figures for the base year and the current year.

Laspeyre's quantity index formula given in the second form is used when we are given the base year's value figures and the quantity figures for the base year and the current year.

* Quantity of a commodity multiplied by its prices is called the value of that commodity, Base year's quantity multiplied by base year's price is called base year's value. Therefore Q_0P_0 or P_0Q_0 represents base year's value of commodity.

Check your Progress

8. Define Laspeyre's price index.

Space for hints

8.3.3 Merit of Laspeyre's Index Number :

Use of arithmetic mean has an upward bias while geometric mean has no bias. But once base year's values are used as weights they introduce a downward bias. Since they introduce a downward bias, the two biases are in opposite directions in the case of weighted mean with base year's values as weights. It has smaller index compared to weighted geometric mean. Hence, Laspeyre's index has smaller error and is used in the construction of a number of leading index numbers.

Example-8 :

Calculate Laspeyre's price index and quantity index from the following data taking 1939 as the base year.

Year	Commodity A		Commodity B	
	Price	Quantity	Price	Quantity
1939	4	1	1	10
1953	10	2	4	25

$$\text{Laspeyre's price index, } P_{01} = \frac{\Sigma Q_0 P_1}{\Sigma Q_0 P_0} \times 100$$

$$\text{Laspeyre's quantity index } Q_{01} = \frac{\Sigma P_0 Q_1}{\Sigma P_0 Q_0} \times 100$$

So we need the values of $\Sigma Q_0 P_1$, $\Sigma Q_0 P_0$ (or $\Sigma P_0 Q_0$) and $\Sigma P_0 Q_1$. To get these values we form the following table.

Commodity	1939		1953		Q_0P_1	Q_0P_0	P_0Q_1
	P_0	Q_0	P_1	Q_1			
A	4	1	10	2	10	4	8
B	1	10	4	25	40	10	25
					$\Sigma Q_0P_1 = 50$	$\Sigma Q_0P_0 = 14$	$\Sigma P_0Q_1 = 33$

$$\text{Laspeyre's price index} = \frac{\Sigma Q_0 P_1}{\Sigma Q_0 P_0} \times 100$$

$$= \frac{50}{14} \times 100$$

$$= 357$$

Space for hints

$$\text{Laspeyre's quantity index} = \frac{\Sigma P_0 Q_1}{\Sigma P_0 Q_0} \times 100$$

$$= \frac{33}{14} \times 100$$

$$= 235.7$$

Example-9 :

Find out Laspeyre's price index from the following data taking 1969 as the base year.

Commodity	1969 Value Rs.	1969 Price Rs.	1970 Price Rs.
Sugar	60	3	4
Flour	80	2	2.5
Milk	10	1	1.5

1969 is the base year.

K value figures given for 1969 are the base year's value figures. That is, they are the values of $Q_0 P_0$.

Price figures 1969 are base year's price figures.

$\therefore$ They are the values of P_0.

1970 is the current year. K Price figures for 1970 are the values of P_1.

The formula to get Laspeyre's price index is

$$P_{01} = \frac{\Sigma Q_0 P_0 \times \frac{P_1}{P_0}}{\Sigma Q_0 P_0} \times 100$$

We get the required values from the following table :

Commodity	$Q_0 P_0$	P_0	P_1	$\frac{P_1}{P_0}$	$Q_0P_0 \times \frac{P_1}{P_0}$
Sugar	60	3	4	$\frac{4}{3}$	$60 \times \frac{4}{3} = 80$
Flour	80	2	2.5	$\frac{2.5}{2}$	$80 \times \frac{2.5}{2} = 100$
Milk	10	1	1.5	$\frac{1.5}{1}$	$10 \times 1.5 = 15$
Total	$\Sigma Q_0P_0 = 150$				$\Sigma Q_0P_0 \times \frac{P_1}{P_0} = 195$

Space for hints

$$\therefore \text{Laspeyre's price index} = \frac{\Sigma Q_0 P_0 \times \frac{P_1}{P_0}}{\Sigma Q_0 P_0} \times 100$$

$$= \frac{195}{150} \times 100$$

$$= 130$$

8.4 Paasche's Index Number

8.4.1 Definition

The weighted aggregate price index with current year's quantities as weights or quantity index with current year's prices as weights is called Paasche's index number. The formulae are,

$$P_{01} = \frac{\Sigma Q_1 P_1}{\Sigma Q_1 P_0} \times 100$$

$$Q_{01} = \frac{\Sigma P_1 Q_1}{\Sigma P_1 Q_0} \times 100$$

So, Paasche's price index can be obtained if the price figures alone for the base year and the price and quantity figures for the current year are given.

Paasche's quantity index can be obtained if the quantity figures alone for the base year and the price and quantity figures for the current year are given.

Example-10 :

Calculate Paasche's index numbers using the same data given in Example-8 and taking 1939 as base.

$$\text{Paasche's price index} = \frac{\Sigma Q_1 P_1}{\Sigma Q_1 P_0} \times 100$$

$$\text{Paasche's quantity index} = \frac{\Sigma P_1 Q_1}{\Sigma P_1 Q_0} \times 100$$

So, we need the values of $\Sigma Q_1 P_1$ (or $\Sigma P_1 Q_1$). $\Sigma Q_1 P_0$ and $\Sigma P_1 Q_0$. To get these values we form the following table :

Commodity	1939		1953		Q_1P_1	Q_1P_0	P_1Q_0
	P_0	Q_0	P_1	Q_1			
A	4	1	10	2	20	8	10
B	1	10	4	25	100	25	40
					$\Sigma Q_1P_1 = 120$	$\Sigma Q_1P_0 = 33$	$\Sigma P_1Q_0 = 50$

$$\therefore \text{Paasche's price index} = \frac{\Sigma Q_1 P_1}{\Sigma Q_1 P_0} \times 100$$

Check your Progress

9. Give the formula for Pasche's price index.

$$= \frac{120}{33} \times 100$$

$$= 363.6$$

Paasche's quantity index $= \frac{\Sigma P_1 Q_1}{\Sigma P_1 Q_0} \times 100$

$$= \frac{120}{50} \times 100$$

$$= 240$$

Space for hints

8.4.2 Alternative expression

Just as Laspeyre's index numbers Paasche's index number can also be rewritten as weighted average of relatives as follows :

$$P_{01} = \frac{\Sigma Q_1 P_1}{\Sigma \frac{Q_1 P_1}{P_1/P_0}} \times 100$$

$$Q_{01} = \frac{\Sigma P_1 Q_1}{\Sigma \frac{P_1 Q_1}{Q_1/Q_0}} \times 100$$

According to these formulae, to get Paasche's price index it is enough if we calculate weighted harmonic mean of price relatives taking current year's value* as weights. To get Paasche's quantity index, it is enough if we calculate the weighted harmonic mean of the quantity relatives taking current year's values as weights.

So, when we are given current year's value figures and price figures for the base year and current year, we use Paasche's price index formula given in the second form.

When we are given current year's value figures and quantity figures for the base year and current year we use Paasche's quantity index formula given in the second form.

Example-11 :

Calculate Paasche's price index from the following data taking 1969 as the base year.

Commodity	1969	1970	
	Price Rs.	Value Rs.	Price Rs.
Sugar	3	88	4
Flour	2	75	2.5
Milk	1	22.5	1.5

* Current year's quantity multiplied by current year's price is called the current year's value. Therefore, Q_1P_1 or P_1Q_1 is the current year's value of a commodity.

Space for hints

1969 is the base year $\therefore$ Price figures given for 1969 are the values of P_0.
1970 is the current year.

$\therefore$ Value figures given for 1970 are the current year's values. That is, they are the values of Q_1P_1.

Price figures given for 1970 are the values of P_1.

The formula to get Paasche's price index is,

$$P_{01} = \frac{\Sigma Q_1P_1}{\Sigma \frac{Q_1P_1}{P_1/P_0}} \times 100$$

$\therefore$ We need the values of ΣQ_1P_1 and $\Sigma \frac{Q_1P_1}{P_1/P_0}$

To get these values, we form the following table :

Commodity	P_0	Q_1P_1	P_1	$\frac{P_1}{P_0}$	$\frac{Q_1P_1}{P_1/P_0}$
Sugar	3	88	4	$\frac{4}{3}$	$\frac{88}{4/3} = 66$
Flour	2	75	2.5	$\frac{2.5}{2}$	$\frac{75}{2.5/2} = 60$
Milk	1	22.5	1.5	$\frac{1.5}{1}$	$\frac{22.5}{1.5/1} = 15$
Total		$\Sigma Q_1P_1 = 185.5$			$\Sigma \frac{Q_1P_1}{P_1/P_0} = 141$

$$\therefore \text{ Paasche's price index} = \frac{\Sigma Q_1P_1}{\Sigma \frac{Q_1P_1}{P_1/P_0}} \times 100$$

$$= \frac{185.5}{141} \times 100$$

$$= 131.5$$

8.4.3 Merit of Paasche's index numbers :

Use of harmonic mean introduces a downward bias and the use of current year's values are weights introduces an upward bias. Since the two biases are in opposite directions here also the net bias is smaller.

8.4.4 Comparison of Paasche's index numbers :

Laspeyre's index is preferred to Paasche's index in many practical cases. The reasons for this are as follows.

(i) In Laspeyre's index, base year's quantities (in the case of price index) and

base year's prices (in the case of quantity index) are used as weights. So when we calculate index numbers, the weights do not change from one year to the next. But, in Paasche's index, current year's quantities (in the case of price index) and current year's prices (in the case of quantity index) are used as weights. So, the weights change from one year to the next year and these new weights can be had only if up to date data are available. Therefore, in the case of Paasche's index number it is often difficult and expensive to obtain the weights.

Space for hints

(ii) Laspeyre's index for any two year's (constructed with the same base) can be compared with each other as they have the same weights. On the other hand, suppose L_{70}, L_{71} and L_{72} are Laspeyre's index numbers for the year 1970, 1971 and 1972 respectively with 1970 as base (i.e., $L_{70} = 100$). Here, we can compare L_{70} with L_{71}. On the other hand suppose P_{70}, P_{71} and P_{72} are Paasche's index numbers for 1970, 1971 and 1972 respectively with 1970 as base (i.e., $P_{70} = 100$). Here we cannot compare P_{71} with P_{72} as they have different weights. We can compare each of P_{71} and P_{72} only with P_{70}.

Thus, Paasche's index suffers from lack of comparability.

Hence, Laspeyre's index remains most popular in practice.

Due to the difference in the weights taken, Laspeyre's and Paasche's formulae used to compute either the price or the quantity index give different results. This difference will be small if no drastic changes have taken place between the base year and the current year.

It is an interesting property to be noted that Laspeyre's index number usually tends to *over estimate* (or in otherwords, it tends to have an upward bias) whereas Paasche's index tends to *under estimate* (or in other words, it tends to have a downward bias). Hence, instead of taking either of the two index numbers to denote the change, we take both the index numbers to give us a meaningful range of the value changed. For example, if the price index computed by one formula is 110 and by the other formula is 115, we may say that the price level has changed from, 100 to a value between, 110 and 115. This statement gives us useful, though not precise, information.

8.5 Fisher's index number :

8.5.1 Definition

Fisher's index number is the geometric mean of Laspeyre's and Paasche's index numbers. Hence,

$$P_{01} = \sqrt{\frac{\Sigma Q_0 P_1}{\Sigma Q_0 P_0} \times \frac{\Sigma Q_1 P_1}{\Sigma Q_1 P_0}} \times 100$$

$$Q_{01} = \sqrt{\frac{\Sigma P_0 Q_1}{\Sigma P_0 Q_0} \times \frac{\Sigma P_1 Q_1}{\Sigma P_1 Q_0}} \times 100$$

Space for hints

8.5.2 Fisher's index number

Fisher's index number is also called Fisher's ideal index number. The reasons for Fisher's index being called 'ideal' are as follows :

(i) To construct index numbers geometric mean is considered to be the best of all the averages. Because, in the problems of index numbers, we want to measure relative changes and not absolute changes and geometric mean alone satisfies this requirement, Fisher's index is based on the geometric mean.

(ii) Both base year and current year prices and quantities are taken into account in Fisher's Index.

(iii) It is free from bias.

(iv) It satisfies all the three tests* of an ideal index number.

8.5.3 Fisher's index is not popular in practice :

Reasons are

(i) It involves laborious calculations.

(ii) Because of the presence of Paasche's component, just like Paasche's index, Fisher's index also suffers from the same difficulty viz, lack of comparability.

Example-12 :

The following table gives the data of production in tons and price per ton of four principal crops in India, during the years 1960-61 and 1969-70. Taking 1960-61 as base, construct Fisher's ideal index number of prices for 1969-70.

Commodity	Production (Lakhs of Tons)		Price (Rupees per Ton)	
	1960-61	1969-70	1960-61	1969-70
A	250	300	150	130
B	100	120	120	200
C	20	30	600	1,000
D	10	20	200	300

Fisher's price Index $P_{01} = \sqrt{\dfrac{\Sigma Q_0P_1}{\Sigma Q_0P_0} \times \dfrac{\Sigma Q_1P_1}{\Sigma Q_1P_0}} \times 100$

1960-61 is the base year and 1969-70 is the current year.

∴ Production figures and price figures for 1960-61 are the values of Q_0 and P_0 respectively.

Check your Progress

10. Which is the ideal price index? Give its formula.

* We will discuss about these three tests in details later in this Unit.

Space for hints

Production figures and price for 1969-70 are the values of Q_1 and P_1 respectively.

Commodity	Q_0	Q_1	P_0	P_1	Q_0P_1	Q_0P_0	Q_1P_1	Q_1P_0
A	250	300	150	130	32500	37500	39000	45000
B	100	120	120	200	20000	12000	24000	14400
C	20	30	600	1000	20000	12000	30000	18000
D	10	20	200	300	3000	2000	6000	4000
Total					75500	63500	99000	81400

$\therefore \Sigma Q_0 P_1 = 75500 \qquad \Sigma Q_0 P_0 = 63500$

$\Sigma Q_1 P_1 = 99000 \qquad \Sigma Q_1 P_0 = 81400$

$$P_{01} = \sqrt{\frac{\Sigma Q_0P_1}{\Sigma Q_0P_0} \times \frac{\Sigma Q_1P_1}{\Sigma Q_1P_0}} \times 100$$

$$= \sqrt{\frac{75500}{63500} \times \frac{99000}{81400}} \times 100$$

$$= \sqrt{\frac{151}{127} \times \frac{495}{407}} \times 100$$

$\log 151 \quad = 2.1790$

$\log 495 \quad = 2.6946$

$\log 151 + \log 495 \quad = 2.1790 + 2.6946 = 4.8736$

$\log 127 \quad = 2.1038$

$\log 407 \quad = 2.6096$

$\log 124 + \log 407 \quad = 2.1038 + 2.6096 = 4.7134$

$(\log 151 + \log 495) - (\log 127 + \log 407) = 4.8736 - 4.7134 = 0.1602$

$$\frac{.1602}{2} = .0801$$

Anti-log .0801 $= 1.202$

$$\therefore \sqrt{\frac{151}{127} \times \frac{495}{407}} = 1.202$$

$\therefore \; P_{01} = 1.202 \times 100 = 120.2$

Fisher's index number of prices for 1969-70 $= 120.2$

Space for hints

Example-13 :

Calculate Fisher's ideal index number from the following table.

Commodities	Price in Rupees		Quantity	
	1955	1956	1955	1956
Rice	20.00	15.00	1	1.25
Salt	4.00	4.75	10	8
Cloth	10.50	12.50	20	18
House rent	10.00	12.00	1	1

Let us take 1955 as the base year and calculate Fisher's index for 1956 (i.e., we take 1956 as the current year).

Fisher's price index is,

$$P_{01} = \sqrt{\frac{\Sigma Q_0 P_1}{\Sigma Q_0 P_0} \times \frac{\Sigma Q_1 P_1}{\Sigma Q_1 P_0}} \times 100$$

Fisher's quantity index is,

$$Q_{01} = \sqrt{\frac{\Sigma P_0 Q_1}{\Sigma P_0 Q_0} \times \frac{\Sigma P_1 Q_1}{\Sigma P_1 Q_0}} \times 100$$

Commodities	P_0	P_1	Q_0	Q_1	Q_0P_1	Q_0P_0	Q_1P_1	Q_1P_0
Rice	20.00	15.00	1	1.25	15.00	20.00	18.75	25.00
Salt	4.00	4.75	10	8	47.50	40.00	38.00	32.00
Cloth	10.50	12.50	20	18	250.00	210.00	225.00	189.00
House rent	10.00	12.00	1	1	12.00	10.00	12.00	10.00
Total					324.50	280.00	293.75	256.00

$\Sigma Q_0 P_1 = 324.50$ $\quad\quad$ $\Sigma Q_1 P_1 = 293.76$

$\Sigma Q_0 P_0 = 280.00$ $\quad\quad$ $\Sigma Q_1 P_0 = 256.00$

$$\therefore P_{01} = \sqrt{\frac{\Sigma Q_0 P_1}{\Sigma Q_0 P_0} \times \frac{\Sigma Q_1 P_1}{\Sigma Q_1 P_0}} \times 100$$

$$= \sqrt{\frac{324.50}{280.00} \times \frac{293.75}{256.00}} \times 100$$

Space for hints

log 324.50	= 2.5112	
log 293.75	= 2.4679	
Total	= 4.9791	

log 280	= 2.4472	
log 256	= 2.4082	
Total	= 4.8554	

4.9791 – 4.8554 = 0.1237

$$\frac{.1237}{2} = .0618$$

Anti-log of .0618 = 1.153

$$\sqrt{\frac{324.50}{280.00} \times \frac{293.75}{256.00}} = 115.3$$

$\therefore$ P_{01} = 1.153 X 100 = 115.3

$$Q_{01} = \sqrt{\frac{256.00}{280.00} \times \frac{293.75}{324.50}} \times 100$$

log 256	= 2.4082
log 293.75	= 2.4679
Total	= 4.8761

log 280	= 2.4472
log 324.5	= 2.5112
Total	= 4.9584

4.8761 – 4.9584 = 4 + .8761 – 4 – .9584

= 4 – 4 – 1 + 1.8761 – .9584

= –1 + 1.8761 – .9584

= –1 + .9177 = –1.9177

Space for hints

$$\frac{\bar{1}.9177}{2} = \frac{-1 + .9177}{2} = \frac{-2 + 1.9177}{2}$$

$$= -1 + .9588 = \bar{1}.9588$$

Anti-log $\bar{1}.9588 = 0.9095$

$$\therefore \sqrt{\frac{256}{280} \times \frac{293.75}{256.00}} = 0.9095$$

$$\therefore Q_{01} = .9095 \times 100 = 90.95$$

$\therefore$ Fisher's price index $= 115.3$

$\therefore$ Fisher's quantity index $= 90.95$

9. Tests of Consistency for an Index Number

Every good index number should satisfy certain tests so that it may be free from bias. The tests are :

(i) Commodity reversal test :

(ii) Time reversal test and

(iii) Factor reversal test.

An index number which satisfies all the three tests given above are said to be ideal or perfect index number.

9.1 Commodity reversal test :

9.1.1 Definition :

Suppose the prices of a group of four commodities say, A, B, C and D are given for two years. Suppose a person, X takes commodities in the order A, B, C, D and calculates the index number. Suppose another person Y takes these commodities in the order B, D, A, C and calculates the index number using the same formula. Now according to the commodity reversal test, both X and Y should get the same value for the index number.* So, we can state the commodity reversal test as follows:

If the order of taking the commodities is changed, the index number should still remain unchanged.

*In our example, if X and Y get different values, then it is absurd and the index number does not represent the reality. So, it is a basic necessity that every index number is to be constructed in such a way that its value remains unchanged even if the order of taking commodities is changed. Thus strictly speaking, commodity reversal test is a basic property of all the index numbers rather than a test of consistency for index numbers.

9.1.2 Which index satisfies the test

This test is satisfied by all the index numbers.

9.2 Time reversal test :

9.2.1 Definition :

Suppose we are given price and quantity data for two years 1960 and 1970. Suppose we take 1960 as base and calculate the price index for 1970 by some formula. Let us denote this index by P_{01}. Now let us interchange the base year and current year and again get the price index by the same formula. Let us denote this second index by P_{01}. Now according to the time reversal test, P_{01} and P_{10} should be reciprocals of each other. In other words, when P_{01} is multiplied by P_{10}, the product should be equal to unity (one). So we can state the time reversal test as follows.

When we use the same formula and get the index numbers for any two years with bases reversed, the two index numbers should be reciprocals of each other. That is, the product of the two index numbers should be equal to unity. Symbolically, we can give this test as follows :

In the case of price index,

$$P_{01} \times P_{10} = 1$$

In the case of Quantity index,

$$Q_{01} \times Q_{10} = 1$$

If for a given index, the product is not equal to 1 then, we say that the time reversal test is not satisfied by that index number.

If any index does not satisfy this test, then it is said to have "type bias" meaning thereby that it does not indicate the real state of facts.

9.2.2 Which Index satisfies the test

Only Fisher's index and Marshall-Edgeworth index satisfy this test. All the other index numbers do not satisfy this test.

9.2.3 Testing Fisher's index

Fisher's price index*

$$P_{01} = \sqrt{\frac{\Sigma Q_0 P_1}{\Sigma Q_0 P_0} \times \frac{\Sigma Q_1 P_1}{\Sigma Q_1 P_0}}$$

* In the case of tests of consistency we take all the index number formulae without multiplying by 100.

Space for hints

Check your Progress

11. What is Commodity Reversal Test?

12. Define Time Reversal Test?

Space for hints

In this formula, wherever we have subscript 0 we change it to 1 and wherever we have the subscript 1 we change it to 0. This will give us P_{10}.

$$P_{10} = \sqrt{\frac{\Sigma Q_1 P_0}{\Sigma Q_1 P_1} \times \frac{\Sigma Q_0 P_0}{\Sigma Q_0 P_1}} \times 100$$

Now,

$$P_{01} \times P_{10} = \sqrt{\frac{\Sigma Q_0 P_1}{\Sigma Q_0 P_0} \times \frac{\Sigma Q_1 P_1}{\Sigma Q_1 P_0}} \times \sqrt{\frac{\Sigma Q_1 P_0}{\Sigma Q_1 P_1} \times \frac{\Sigma Q_0 P_0}{\Sigma Q_0 P_1}}$$

$$= \sqrt{\frac{\Sigma Q_0 P_1}{\Sigma Q_0 P_0} \times \frac{\Sigma Q_1 P_1}{\Sigma Q_1 P_0} \times \frac{\Sigma Q_1 P_0}{\Sigma Q_1 P_1} \times \frac{\Sigma Q_0 P_0}{\Sigma Q_0 P_1}}$$

$$= 1$$

$\therefore$ Fisher's price index satisfies Time Reversal Test.

6.2.4 Testing Laspeyre's Index

$$P_{01} = \frac{\Sigma Q_0 P_1}{\Sigma Q_0 P_0}$$

$$\therefore \ P_{10} = \frac{\Sigma Q_1 P_0}{\Sigma Q_1 P_1}$$

$\therefore \ P_{01} \times P_{10} = \dfrac{\Sigma Q_0 P_1}{\Sigma Q_0 P_0} \times \dfrac{\Sigma Q_1 P_0}{\Sigma Q_1 P_1}$ which is not equal to 1 because nothing gets cancelled out.

$\therefore$ Laspeyre's index does not satisfy time reversal test.

9.2.5 Testing Paasche's Index

Paasche's Price Index

$$P_{01} = \frac{\Sigma Q_1 P_1}{\Sigma Q_1 P_0}$$

$$\therefore \ P_{10} = \frac{\Sigma Q_0 P_0}{\Sigma Q_0 P_1}$$

$P_{01} \text{ X } P_{10} = \dfrac{\Sigma Q_1 P_1}{\Sigma Q_1 P_0} \times \dfrac{\Sigma Q_0 P_0}{\Sigma Q_0 P_1}$ which is not equal to 1.

$\therefore$ Paasche's index also does not satisfy time reversal test.

$\therefore$ Paasche's index has 'type bias'.

9.3 Factor Reversal Test :

9.3.1 Definition

This test defines that the formula for an index of change of prices should serve equally well as that for an index of changes in quantities. In other words if P and Q are inter-changed (the subscripts remaining unaltered) in a price index it should yield the quantity index; also the product of the price and quantity indices should give the value ratio, $\frac{\Sigma Q_1 P_1}{\Sigma Q_0 P_0}$

Thus, if P_{01} and Q_{01} are the price and quantity indices respectively, according to Factor Reversal Test.

$$P_{01} \text{ X } Q_{01} = \frac{\Sigma Q_1 P_1}{\Sigma Q_0 P_0}$$

9.3.2 Which index satisfies the test

This test is satisfied in the case of a single commodity. It is explained below :

In the case of a single commodity.

$$P_{01} = \frac{P_1}{P_0}$$

$$Q_{01} = \frac{Q_1}{Q_0}$$

$$\therefore P_{01} \times Q_{10} = \frac{P_1 Q_1}{P_0 Q_0} \text{ which is the value ratio}$$

But the test requires that it must be satisfied when a number of commodities are given.

In the case of a number of commodities.

This test is satisfied only by Fisher's index number. All the other index numbers do not satisfy factor reversal test.

9.3.3 Testing Fisher's Index

Let us see how Fisher's index satisfies factor reversal test. Fisher's price index is

$$P_{01} = \sqrt{\frac{\Sigma Q_0 P_1}{\Sigma Q_0 P_0} \times \frac{\Sigma Q_1 P_1}{\Sigma Q_1 P_0}} \times 100$$

Putting Q wherever we have the letter P and putting P wherever we have the letter Q in the above price index number,

$$Q_{01} = \sqrt{\frac{\Sigma P_0 Q_1}{\Sigma P_0 Q_0} \times \frac{\Sigma P_1 Q_1}{\Sigma P_1 Q_0}} \times 100$$

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Check your Progress

13. Define Factor Reversal Test.

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Now,

$$P_{01} \times Q_{10} = \sqrt{\frac{\Sigma Q_0P_1}{\Sigma Q_0P_0} \times \frac{\Sigma Q_1P_1}{\Sigma Q_1P_0}} \times \sqrt{\frac{\Sigma P_1Q_0}{\Sigma P_1Q_1} \times \frac{\Sigma P_0Q_0}{\Sigma P_0Q_1}}$$

$$= \sqrt{\frac{(\Sigma Q_1P_1)\,(\Sigma P_1Q_1)}{(\Sigma Q_0P_0)\,(\Sigma P_0Q_0)}}$$

$$= \frac{\Sigma P_1Q_1}{\Sigma P_0Q_0} = \text{value ratio}$$

As Fisher's index satisfies both the time and factor reversal tests, it is said to be an ideal index number.

9.3.4 Testing Laspeyre's price index,

$$P_{01} = \frac{\Sigma Q_0P_1}{\Sigma Q_0P_0}$$

$$Q_{01} = \frac{\Sigma P_0Q_1}{\Sigma P_0Q_0}$$

$\therefore\ P_{01} \times Q_{01} = \dfrac{\Sigma Q_0P_1}{\Sigma Q_0P_0} \times \dfrac{\Sigma P_0Q_1}{\Sigma P_0Q_0}$ which is not equal to the value ratio viz, $\dfrac{\Sigma P_1Q_1}{\Sigma P_0Q_1}$

$\therefore$ Laspeyre's index does not satisfy factor reversal test.

9.3.5 Testing Paasche's Index

Paasche's Price Index

$$P_{01} = \frac{\Sigma Q_1P_1}{\Sigma Q_1\,P_0}$$

$$Q_{01} = \frac{\Sigma P_1Q_1}{\Sigma P_1Q_0}$$

$$\therefore\ P_{01} \times Q_{01} = \frac{\Sigma Q_1P_1}{\Sigma Q_1P_0} \times \frac{\Sigma P_1Q_1}{\Sigma P_1Q_0}$$

Here also the product is not equal to $\dfrac{\Sigma P_1Q_1}{P_0Q_0}$

$\therefore$ Paasche's index also does not satisfy factor reversal test

9.4 The Circular Test :

This test is an extension of the time reversal test. The time reversal test takes into account only two years, the current and base years. The circular test requires this property to hold good for any two years.

An index number is said to satisfy the circular test if

$$P_{12} \times P_{23} \times P_{34} \times P_{45} \ldots\ldots \times P_{(n-1)n} \times Pn_1 = 1$$

P_{12} is the index of the second year with first year as base.

P_{23}	,,	,,	,,	third	,,	,, second	,,	,,
P_{34}	,,	,,	,,	fourth	,,	,, third	,,	,,
P_{45}	,,	,,	,,	fifth	,,	,, fourth	,,	,,
....								
....								
$P_{(n-1)n}$	,,	,,	,,	nth	,,	,, (n–1)th	,,	,,
Pn_1	,,	,,	,,	first	,,	,, nth	,,	,,

This test is satisfied only in the case of single commodity. Fisher's ideal index satisfies this test only approximately. But the ideal formula is specially meant for comparison of two specific years rather than a series of years. Hence, the circular test is less important for it.

10. Construction of Wholesale Price Index Number

The study of price changes is a very important one in the modern economic set up as it affects practically all branches of economic activity. The changes in the general price level are studied with the help of wholesale price index numbers. In constructing wholesale price index numbers the following factors are to be taken into account.

1. The selection of items, their number and obtaining price quotations.
2. The selection of the base year
3. The selection of a suitable average.
4. The selection of suitable weights.
5. The selection of suitable formula.

In addition to the above five factors the object of the index numbers is important. All the five factors given above are affected considerably by the purpose for which a particular index is constructed.

For measuring the general tendency of the price level, items are to be selected out of a very large number. If we are interested in knowing the general tendency of the price of agricultural commodities only, the selection of items would be from a

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Check your Progress

14. What is Circular Test?

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smaller number. To study the changes in the general price level during the war period, the base year should be the year immediately preceding the war period. To study the changes in the general price level in the post war period the year immediately succeeding the war period should be taken as the base year. If an index is constructed for food articles weightage given to each food article should be high compared to its weight in a general purpose index number of wholesale prices. Thus we arrive at a very important conclusion that in the construction of an index number the various problems involved should be viewed in the light of the purpose for which a particular index number is prepared.

Now let us study the five factors mentioned above, in that very order.

10.1 Selection of items, their number and price quotation :

i) Selection of items

Firstly, the commodities are divided into different classes. Then one or two commodities from each class are selected in such a manner that they will adequately represent their class. Such selected commodities may be called representative commodities. The representative commodities should have the following two characteristics.

(a) They should adequately represent the tastes, habits, customs and necessities of the people. Wholesale price index numbers are constructed to get an idea about the changes in the general price level which would help to make a proper study of the effects of price changes on various economic and social problems. If selected commodities are not representatives of the tastes, customs, habits and necessities of society, the conclusions derived from the index number would never be applicable to the problems of the people for whom it is meant. For instance, a wholesale price index number including foreign wines, heavy machines, refrigerators, costly automobiles, etc., would not represent the price changes correctly because the changes in the price of these commodities hardly matter much in the case of majority of Indians.

(b) The selected commodities should be stable in quality and preferably should be standardized or graded. If the selected commodity is not stable in quality, each time when its price is obtained it would technically refer to the price of something different from the one whose price was obtained in the previous time. Hence, the comparison of the relative price level will not be possible.

(ii) Number of items :

There is no rigid rule regarding the number of items of commodities to be selected. Theoretically, larger the number of items, the more accurate the index number would be. But a large number of items involves difficulty in calculation. So,

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the number should he very large consistent with the ease in handling them. In India, the Economic Adviser's office publishes general purpose wholesale price index by taking into account 360 commodities. In other countries a large number of items are taken into account.

Ordinarily all those varieties which are in common use should be included. If the price of these varieties are averaged before their inclusion in the index number, the commodity in question does not receive any extra weightage; otherwise, this commodity receives a weightage equal to the number of varieties included. Calcutta wholesale price index number and Bombay wholesale price index number used to give weights to various commodities in this fashion.

(iii) Classification of items :

To have an accurate index number it is better to classify the given items into certain homogeneous groups and find an index number for each of those groups. This helps in studying separately the price variations of different groups. For instance, the Economic Adviser's Index of wholesale price classifies the items into the following three major groups :

I. Primary articles

II. Fuel, Power, Light and Lubricants

III. Manufactured products

These three major groups are further classified into various groups. For instance, primary articles are divided into three groups namely food articles, non-food articles and minerals. The total number of groups is 14.

Further subdivision is made in the case of each of these groups. For example food articles are divided into seven sub-groups viz., cereals, pulses, fruits and vegetables, milk and milk products, eggs fish and meat and other food articles. The total number of sub-groups is 21.

(iv) Obtaining Price Quotations :

After the selection and classification of the commodities, the next problem is the collection of their prices. Just as all commodities cannot be included, price quotation from all places where a commodity is purchased or sold cannot be obtained. Therefore, firstly, some representative places and persons have to be selected. Generally such places are chosen where a particular commodity is purchased or sold in large quantities. Nextly some representatives are appointed to supply the price quotations from time to time. This work can be done either by appointing special staff or by giving the work to some selected individuals or institutions of that particular locality. The information published in journals or magazines about the price ruling in various places can also be utilized. To keep a check it is always better to appoint more than one person preferably three or four in each selected locality.

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The price can be quoted in two ways :

a) Commodity price which is given as quantity of commodity per unit of money. For example 2 bananas for one rupee is commodity price. b) Money price which is given as the quantity of money per unit of commodity. For example : one banana for 50 paise is money price.

In the construction of wholesale price index number prices should be money prices. Also these money prices should be wholesale prices and not retail prices. The reason for this is that wholesale prices are more stable than retail prices. Besides this, wholesale prices are more quickly affected by changes in demand or supply or by other similar factors than retail prices. Retail prices depend upon wholesale prices.

If the price of a commodity is controlled, then control price is taken into account even though the price in the black market may be much higher.

The price quotations have to be obtained per week or per month as the case may be. Generally, larger the number of quotations the better it is. But, too many quotations also complicate the problem or construction of index numbers. Ordinarily, if an index number is constructed every week, one quotation per week is considered sufficient.

The number of quotations should be such that agency supplying the quotation can easily and regularly send them. In the absence of actual quotations prices have to be estimated and the index number becomes comparatively inaccurate.

If an index number is to be published monthly and if four quotations per week from each place are obtained from 20 places, then every week we will have 80 quotations for each commodity. Average of these eighty quotations gives the average weekly price of commodity. Four such averages are calculated for each week. These four weekly averages are again averaged and the resultant is monthly average price of the commodity for the whole country. This monthly averaged price would be used in the construction of the index number.

10.2 Selection of the base year :

After the prices have been collected, the next step in the construction of index number is to reduce them to percentages or relatives. For this an appropriate base in terms of which the prices shall be expressed as percentages should be selected. There are two methods by which the base can be selected. (i) Fixed base and (ii) Chain base.

(i) Fixed base method :

In the fixed base method either (i) the price of some arbitrarily chosen year or (ii) the averages of the prices of a period consisting of three, five or even ten years is taken as the base. The base so chosen can be used for an indefinite period.

The year chosen as the base should be a normal year. In other words, the price level in this year should neither be abnormally low nor abnormally high. If the year chosen happens to be an abnormal year for instance a year of labour unrest or of war or of financial crisis, the index number would give misleading conclusions. To avoid this, the average price of a period of a few years is taken as the base price.

(ii) Chain base method :

In this method the base year is not fixed, but changes from year to year. For each year the previous year is taken as the base year. The chief advantage of this method is that price relative of a year can be compared with the price level of the immediately preceding year. Businessmen and others are more interested in comparison of this type rather than in comparison relating to the distant past. Another advantage is that it is possible to include new items or to delete old items which are no more important.

The drawback of this method is that with it comparisons cannot be made over a long period.

10.3 Selection of suitable average :

When the price changes of only one commodity have to be studied the simple percentage price relatives are the relevant index numbers. Here the problems of choice of average does not rise. But, in general, we want to study changes in the general price level. In such cases more than one commodity have to be taken into account. Average of the price relatives of all such commodities gives the index number of the general price level. Theoretically, the average can be used for the purpose. But practically a choice has to be made between arithmetic mean and geometric mean.

Simplicity of calculation may recommend arithmetic mean. But it does not satisfy the time reversal test.

Geometric mean no doubt suffers from the drawback of difficulty of calculation. But it has certain definite advantages that out-weigh its disadvantage. The advantages are as follows :

i) We are interested in finding out the relative changes in phenomenon using index numbers. Geometric mean measures relative changes and hence it receives preference over averages.

ii) It satisfies time reversal test and hence, does not have any bias.

iii) New items can be added and old ones dropped from the list at any time without having to repeat the calculation's over again.

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Therefore, in the construction of index number invariably geometric mean should be used. However, the Economic Adviser's Index Number of Wholesale Prices in India uses the average weighted arithmetic mean.

10.4 Selection of suitable weights :

When the relative importance of the items is not equal, weighted average gives better results than unweighted average. The weights should not be arbitrary. Depending upon the two important aspects viz, the purpose of the index number and the nature of the data concerned with it, weights should be chosen. For example, to study change in the money income of farmers the data should be farm prices and the weights should be proportional to the total money received from several products.

When we take fixed base year the weight attached to each of them should also be fixed. On the other hand, if chain base is taken, each year different weight is attached to each item.

Index numbers can be weighted in two-ways (i) Implicit weighting and (ii) Explicit weighting.

In the case of explicit weighting, the commodity to which greater importance is attached is repeated a number of times. For example, rice is to receive a weight of three then three varieties of rice will be included.

In the case of explicit weighting, some outward evidence of importance of the various items in the index is given.

The decision about the appropriate explicit weights depend upon answers to the following questions :

(i) By what do we mean weight? What type of weight is to be used? The items from which weights should be taken.

(ii) The weights can be production figures, consumption figures or distribution figures. The choice would depend upon whatever seems appropriate to bring out the economic importance of the commodities involved from the point of our object.

(iii) There are two types of weights viz., quantity weights and value weights.

Quantity weights represent the amount of commodity produced, consumed or distributed.

Value weight is got by multiplying price with quantity produced, consumed or distributed.

Quantity weights are to be used if we adopt method of aggregates because the product of price and quantity will always be in rupees.

On the other hand, quantity weight cannot be used if we adopt the method of price relatives. Because, the product of quantity and price relatives would be in different units. For example, kilograms multiplied by price relatives would give kilograms, meters multiplied by price relatives would give metres and so on. This being so, the product of quantities and price relatives cannot be added and averaged. Because of this difficulty, whenever we adopt the price relatives method, only value weights (which are always expressed in rupees) are used,so that the products are always in rupees.

In short, when the method of aggregate of actual prices is adopted quantity weights are used; when the method of average of price relatives is adopted value weights are used.

(ii) The quantity weights may be the quantities of base period (Q_0) or of the current period (Q_1) or it may be the sum (Q_0+Q_1) or average of these two $\left(\frac{Q_1+Q_0}{2}\right)$

If the quantity of each commodity changed from year to year in the same proportion, the results would be identical whether we use base year quantities or current year quantities as weights. If it is not so and the relative importance of various commodities changes along with changes in their relative prices, the results would be influenced by the selection of the period from which weights are drawn.

When base year quantities are taken as weights, the index would usually, have an upward bias. The reason is as follows. There is a tacit assumption in this method that the current year quantity figures are the same as in the base year irrespective of the changes that have taken place in the prices between the two years. But ordinarily, if the price of a commodity has risen the quantity purchased would less and vice versa. As we have not considered this aspect in this method, greater weight is assigned to commodities for which the price have risen and as such the resultant index is likely to have an upward bias.

When the current year quantities are taken as the weights, resultant index has a downward bias. This is because undue weight has been given to those commodities whose prices have fallen in the current year.

Only because of the above facts, Laspeyres' index always tends to have an upward bias whereas Paasche's index tends to have a downward bias.

Value weights are usually the product of price and quantity of the base year (P_0Q_0) or of the current year (P_1Q_1). Sometimes these weights are obtained by the product of price in any time and quantity in any other time.

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10.5 Selection of suitable formula :

There are a large number of formula for constructing the index number. The problem is to select the suitable formula. The choice of a formula depends on the purpose of the index, the data availability and the accuracy desired. Fisher has suggested that an appropriate index is that which satisfies time reversal test and factor reversal test. Theoretically, Fisher's index is considered as 'ideal' for constructing index numbers. But usually Fisher's index is not used in practice for the following reasons.

(1) It involves laborious calculation

(2) The data, particularly for the Paasche segment of the Index are not readily available.

No one particular formula can be regarded as the best under all circumstances. On the basis of this knowledge of the characteristics of different formula the investigator has to choose the formula suitable to his data and appropriate to his purpose.

11. Usefulness of Index Numbers

Index numbers are indispensable tools of economic and business analysis. Their significance can be best appreciated by the following points :

1. They provide a single method of comparing changes from time to time or from place to place. We can easily compare Rs.2-50 the price of rice per kilo with Rs.1-50 the price of wheat per kilo on a particular day. But, it is not so easy to compare the changes in the prices of these two articles over a period of time. Index numbers of the rice and wheat prices would indicate the relative changes in each price from some given price and which of the two prices had shown a greater change. As the number of items increases this advantage becomes even more apparent.

It is too difficult to compare price difference among several places. By the use of the consumer Price Index number of the Labour Bureau the price differences among various cities can be compared.

2. Index numbers facilitate comparison of changes in series of data expressed in different units eg. rupees, quintals, tonnes or gallons.

3. Index numbers conceal the absolute values and show only relative values. So, whenever firms or other institutions are not willing to reveal the absolute values, Index numbers are useful as they safeguard the identity of data by showing only relative values.

4. The advantage of index numbers is that they are useful in measuring the

relative changes in the values of such variables like general price level, business activity, cost of living etc. which cannot be measured directly.

5. Price indices indicate the general price levels of the country. This enables the Government to follow a correct financial policy.

6. Index number of prices viz., consumer goods are particularly useful to employers, employees, trade union leaders and the government to decide the dearness allowance and such other allowances. Even the Pay Commission of the Government of India has based its recommendations on such index numbers.

7. It is the basic traditional function of a price index to measure the value of money as value of money is its purchasing power and the purchasing power of money is determined by the prices of various commodities. Thus,

$$\text{Value of money} = \text{Purchasing power of money}$$

$$= \frac{1}{\text{Price index}}$$

8. The cost of living indices give an idea of the real wages earned by the workers or by the middle class society. The per capita income of a country studied along with these indices enable us to find out if the standard of living is rising of falling. Wages in factories are often based on cost of living indices.

9. Indices of economic condition like general price level, industrial production, employment, gross national product, etc., are helpful in the study of business conditions and in making business forecasts. Just like an ordinary barometer measures changes in atmospheric pressure and helps to make weather forecasts, the index numbers measure changes in general economic condition and are helpful in making forecasts of business.

They are used to feel the pulse of the economy. Newspapers headline the fact that prices are going up or down, that agricultural production is rising or falling, the sales and profits are higher or lower than in a previous period, as revealed by index numbers. Therefore, index numbers are described as *Economic barometers*. What the barometers reveal in the natural environment, the same is done by the index numbers in the fields of economics and business. If one wants to get an idea as to what is happening to an economy he sh ɔu look at important indices like the index number of industrial production, agricultural production, business activity, etc.

10. Index numbers are useful not only for economists but also in other fields. For example, sociologists measure intelligence quotients which are essentially index numbers comparing a person's intelligence score with that of an average for his age.

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Health authorities prepare indices to display changes in the adequacy of hospital facilities.

11. Since the index numbers study the relative changes in the level of a phenomenon at different periods of time, they are specially useful for the study of the general trend for a group phenomenon in a time series data.

12. Limitations of index Numbers

1) Generally index numbers are based on a sample. While taking a sample random sampling is rarely used. For, taking a random sample from millions of commodities will neither be practical nor be representative. So judgement sample is usually taken and this introduces errors. Efforts must be taken to minimise these errors.

2) As years pass on, qualities of the commodities change. It is very often difficult to take into account of these quality changes. Tastes and habits of people also change with passage of time and they discard certain commodities and begin to use certain other new commodities. An index is a really typical index, only when there is change in the qualities of a commodity. If there is a change in the quality of a commodity its prices in various years would mean prices of different varieties of the same commodity. Because of these factors we cannot make reliable comparisons over long periods.

3) There are a large number of formulae to calculate index numbers. No single formula would give an index which is most satisfactory from all points of view. So we have to select an appropriate formula which will suit our purpose, we meet with certain problems in the selection of an appropriate formula.

4) Index numbers are based on actual data. So lack of adequate and accurate data becomes a serious limitation of the index itself.

5) An all regional or general index may show an error when they are actually applied to regions within the country. For example, the high cost of housing in Bombay makes living costs much dearer than those in Madurai; yet the index of retail prices gives an equal weight for housing for the whole country.

6) It is very difficult to select a base year which is normal in all respects. If a reasonably normal period is not selected, comparison may lead to wrong conclusion.

7) Index numbers are specialized averages and as such they also suffer from weaknesses common to most averages ; it tells us nothing for example, about prices of particular commodities.

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8) Many times, the units of comparison are different, as that the same data may yield different index number at the hand of different individuals.

9) There is no all purpose index number which is applicable for the phenomena. For example, wholesale price index fails to give an idea of the change in the cost of living.

13. Answers to Check Your Progress Questions

1.	Refer 1	8.	Refer 8.3.1
2.	Refer 4.5	9.	Refer 8.4.1
3.	Refer 4.6	10.	Refer 8.5
4.	Refer 7.1	11.	Refer 9.1.1
5.	Refer 7.2	12.	Refer 9.2.1
6.	Refer 8	13.	Refer 9.3
7.	Refer 8.2	14.	Refer 9.4

14. Model questions for guidance:

10 Marks Questions (One Page Answer)

1. What is meant by an index number?
2. What is an ideal index number?
3. What are the merits and demerits of index number?
4. What are the uses of index numbers?
5. Compute Laspere's and Paasche's index numbers from the following data.

Commodity	1965		1975	
	Price Rs.	Quantity Kg.	Price Rs.	Quantity Kg.
A	2	8	4	6
B	5	10	6	5
C	4	14	5	10
D	2	19	2	13

20 Marks Questions (Three Page Answer)

1. Discuss the problems in the construction of index numbers?
2. Mention the test of adequacy of good index numbers and explain any two of them.

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3. Compute Laspere's and Paasche's and Fisher's index numbers.

Commodity	2003		2004	
	p_0	q_0	p_0	q_0
A	8	10	10	12
B	10	12	12	8
C	5	8	5	10
D	4	14	3	20
E	20	5	25	6

4. From the following data given below compute Paasche's and Fisher's index numbers for the year 2004 with the base year 2000.

Commodity	2000		2004	
	Price Rs.	Quantity Kg.	Price Rs.	Quantity Kg.
A	5	2	10	3
B	3	10	6	25
C	20	3	60	4
D	10	6	15	20

5. Explain the types and uses of index number. What are the steps in the construction of it?

6. Write short notes on :

a) Laspeyre's index number

b) Paasche's index number

c) Factor Reversal Test

d) Time Reversal Test.

UNIT - 7

TIME SERIES

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Introduction:

Statistical data may be of two types namely, cross section data and time series data. Data related to the production of paddy in different states of India during the year 2009 or population of world countries in the year 2010 or the heights of students studying I B.A. (Economics) currently through DDE, MKU are all examples cross section data. Here, you have to look at the point that the data in each case represent values of a variable across places or persons at a give point of time. On the other hand, suppose we consider the annual production of paddy in Tamilnadu state alone over the past ten years or the decadal population of India during 1901 and 2001 or the total number of students enrolled for B.A. Economics through DDE, MKU during the years from 1991-2009. In all these cases, data related to different time points noted at equal intervals of time. Such data are called Time Series data. What factors influence the values found in a Time Series? How are Time Series data useful in practice? All such aspects are discussed in a detailed manner in this Unit-7.

Unit Objectives :

After studying this Unit, you would be able to understand

* the meaning of Time Series
* the various components of Time Series
* the methods of measuring the trend of a Time Series

Unit Structure :

1. Time Series - Meaning :
2. Factors Infuencing Time Series
3. Components of Time Series
4. Time Series Models :
5. Secular trend :
6. Seasonal Variations :
7. Cyclical Variations :
8. Irregular or erratic or random fluctuations
9. A comparison of trend, seasonal, cyclical and random fluctuations :

10 Measurement of Secular Trend

11. Answers to Check Your Progress Questions
12. Model questions for guidance

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1. Time Series - Meaning :

A time series is a set of quantitative readings of some variable recorded at equal intervals of time. The interval may be an hour, a day, a week, a month, a year or a group of years. Hourly temperature readings taken at a place on a particular day, daily collections in a cinema theatre for a month, weekly sales figures of a shop, monthly production figures of an industry, yearly import figures of a country, and population figures of a country over a series of census years are all examples of time series.

2. Factors Infuencing Time Series :

Time series are usually affected by a large variety of forces. Among these forces, some are recurring at regular intervals, some others are recurring but not at regular intervals and still others are not at all recurring. The non-recurring forces are said to be random in nature.

3. Components of Time Series

On the basis of the above classification of the forces affecting time series the variations of the time series are usually decomposed into the following four categories.

(i) Secular trend

(ii) Seasonal variations

(iii) Cyclical fluctuations

(iv) Irregular, erratic or random fluctuations.

The above four categories are called the components of time series. Each of the above components represents a well defined type of change in a time series. The job of the investigator is to decompose the series into its four components, make a study of each of these components separately and make inferences about the future. This is called the 'analysis' of time series'.

4. Time Series Models :

Analysis of time series requires decomposition of a series. To decompose a series we must assume that some type of relationship exists among the four components contained in it. Generally we assure either of two types of relationship namely, additive and multiplicative.

Check your Progress

1. What is Time Series?

2. What are the components of Time Series?

4.1 Additive Model

If we assume that the sum of four components gives the original series then we get.

$Y = T + S + C + I$

where Y represents value of the original time series

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T represents trend value

S represents value of the seasonal variation

C represents value of the cyclical variation

I represents value of the irregular fluctuations

The above equation viz., $Y = T + S + C + I$ is called Additive Model.

Features of Additive Model :

(i) All the components are expressed in the units in which the given series is.

(ii) All the components are treated as residuals.

That is,

$(Y - T) = S + C + I$

$(Y - T - S) = C + I$

$(Y - T - S) = I$

(iii) It is assumed that each component does not affect and it is not affected by the value of other components.

4.2 Multiplicative Model :

If it is assumed that the original series is equal to the product of the four components. then

$Y = T \times S \times C \times I$

This is called multiplicative model.

Functions of the multiplicative model:

(i) Trend alone is in the units of the original series. Other components are expressed as relatives or percentages whose average value is 100.

(ii) It is assumed that there is mutual dependence among the components.

(iii) Multiplicative model is mostly used in practice rather than additive model.

5. Secular Trend :

5.1 Definition :

Secular trend may be defined as that component of a time series which reveals the *general tendency of the data over a long period.* Hence, it is also known as long period or long term trend.

5.2 Long Period - Meaning

When we say that secular trend refers to the general tendency of the data over a long period, we have to give the meaning of long period. Ordinarily, one may think

Check your Progress

3. Define Trend.

Space for hints

that long period denotes several years. But it is not so. Here, by long period we mean a period long enough for us to detect uniform change in the data given. Hence, long period does not denote a fixed period of time. In one case even few days may be considered as long period and in another case several years may be considered as long period. Depending upon the nature of data only we have to decide about the long period.

For example, suppose we are counting the bacterial population of a culture for every 5 minutes of a few days and the population continue to increase fairly regularly in those days. Now we can call these few days as long period. Thus in one case few days may constitute a long period of time.

Again, suppose we are given annual production figures for two successive years and in these two years production was on the increase. From this we cannot conclude that there is a general tendency for the production to increase. Only if we are given production figure for several years it is possible for us to assess the secular trend. Thus in another case several years constitute a long period of time.

Hence we come to the conclusion that *long period is not a fixed period* and it is determined by the nature of data given.

5.3 Types of Trend

The secular trend can be either upward or downward. It cannot be bothways. But there are certain exceptions. Some series are such that values fluctuate around a more or less constant figure which does not change with the passage of time. Series relating to the temperature of a particular locality, is an example of such series. The temperature would no doubt fluctutate in various seasons but it would hardly change with the passage of time.

5.4 Factors Affecting Long Trend :

Long term trend reflects the effect of forces which operate over a long period time and are not subject to sudden reversals in direction. Such forces are changes in population or tastes and habits of people etc., For example, the effect of increase in population cannot be sudden or irregular. It would always be very low, gradual and regular.

5.5 Purposes of measuring Trend :

Trend values are measured generally for three principal purposes.

(i) To study the basic growth tendency of series, ignoring short-term fluctuations due to business cycles, seasons, wars or other causes.

(ii) To project the curve as a long term forecast. If the past growth has been

steady and if the conditions that determine this growth may reasonably be expected to persist, a trend curve may be projected over 5 to 10 years into the future as a preliminary forecast. This is the most importantpurpose for measuring secular trend.

(iii) To eliminate it, in order to clarify the cycles and other short term movements in the data.

5.6 Methods of measuring trend :

Trend values are measured generally by any one of the following three methods.

i) Free hand method

ii) Moving average method and

iii) Method of least squares.

We discuss about these methods one by one in the succeeding topics.

6. Seasonal Variations :

6.1 Meaning

Although the word 'seasonal' is synonymous with the season of the year, the term is to denote any periodic movement with a fixed period of a year or less than a year and which repeats itself during every period. Generally, seasonal variations appear at weekly or monthly or quarterly intervals.

6.2 Factors Influencing Seasonal Variations :

Two principal factors are responsible for seasonal changes in a time series. They are (i) change in climate or weather and (ii) customs.

Changes in climate and weather have profound influence in causing seasonal variations. Climatic conditions including variation in rainfall, sunshine, heat and wind produce variations in agricultural productions. Crops can be harvested only at certain times of the year. Soon after the harvesting season prices fall down, and gradually rise to a peak just before the sowing time. Certain types of construction work can be undertaken only when weather condition are favourable. Consumer needs vary widely from one season to another. The consumption of electricity is greater in summer than in winter due to the use of fans and air conditioners.

Although nature is the chief factor governing seasonal variations, customs, habits and traditions also have a considerable effect. For instance, the custom of wearing new clothes on Deepavali and Christmas days causes a marked peak in retail sales of clothes. During the first week of every month bank deposits and daily sales in retail shops are at the highest level.

6.3 Features of Seasonal Variations :

Two important features of the seasonal variations, are (i) they recur in a year

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Check your Progress

4. What are seasonal variations.

Space for hints

with a fixed period and (ii) the increases and decreases occur at about the same time and in about the same proportion in each year.

6.4 Purposes of measuring seasonal movements :

There are three principal purposes of measuring seasonal variation (i) to analyse current seasonal behaviour (ii) to predict seasonal movements as aid in short-term planning and (iii) to eliminate seasonality in order to reveal cyclical movements.

6.5 Methods of measuring seasonal variations :

There are many methods of measuring seasonable variations. Two methods are popularly used in practice :

i) Method of simple averages and

ii) Method of moving averages.

7. Cyclical Variations :

7.1 Meaning :

Cyclical variations are also recurring type of change like seasonal variations. However, they are different from seasonal variations in that they do not have a fixed period. Cyclical fluctuations are also known as business cycles. "Business cycles are a type of fluctuation found in the aggregate economic activity of nations that organize their work mainly in business enterprises: a cycle consists of expansions occuring at about the same time in many economic activities, followed by similar general recessions, contraction and revivals which merge into the expansion phase of the next cycle; this sequence of changes is recurring but not periodic in duration business cycles vary, from more than one year to ten or twelve years".

7.2 Factors contributing to Cyclical variations

Business cycles have developed in modern industrialized countries having closely integrated business structures. The cycles are affected by factors outside business, such as wars, acts of government and the size of crops etc. But, the protracted expansion which gives way to contraction and vice versa in a roughly rhythmic fashion are caused by the conditions within the business itself. Nearly all economic activities are affected by cyclical forces. Heavy industrial production and finance are most susceptible to cyclical forces, and retail trade, personal service and agricultural production are least affected.

7.3 Reasons for measuring cycles :

Two important purposes are served by isolating the cyclical component in a time series.

Check your Progress

5. What are cyclical variations?

6. How do cyclical variations differ from sesonal variations?

Space for hints

(i) Measures of typical cyclical behaviour are valuable aids in controlling the operations of a business. The study of business cycles is also one of the major branches of economics. Today economists generally recognize the need not only of theory but also of accurate statistical measures in order to gain a clear understanding of this phenomenon.

(ii) Successful businessmen plan ahead; planning requires forecasting; and forecasting involves a knowledge of both typical and recent cyclical behaviour.

7.4 Features :

Despite the importance of business cycles they are the most difficult type of economic fluctuation to predict. This is because cycles vary so widely in timing, amplitude (percentage rise and fall) and pattern, and because cyclical variations are in extricably mixed with irregular fluctuations.

8. Irregular or erratic or random fluctuations :

8.1 Meaning and factors contributing to it :

Irregular fluctuations in economic time series are caused by such forces as government expenditures, taxes, unusual weather, strike, war, and all forms of unpredictable events. These, forces are of two types. The first group serves as "originating forces" in inducing or altering business cycle movements. War and its aftermath for example, tend to produce the familiar boom and depression phases of a major peace time cycle.

The second group of irregular factors are usually numerous, unidentifiable and unpredictable and they act in a more or less random fashion.

8.2 Difficult to measure it :

The irregular component in a time series represents the residue of fluctuations after secular trend, cyclical and seasonal movement have been accounted for. In practice, however, the cycle itself is so erratic and is so interwoven with irregular movements that it is impossible to separate them, except in smoothing out some of the random factors of the second type.

9. A comparison of trend, seasonal, cyclical and random fluctuations :

Secular trend denotes gradual growth (or decline) over a long period of time. But seasonal variations represent movements over a period less than one year; cyclical fluctuations are movements over a period greater than one year, usually, ranging between 5 years and 10 years. There is no regular period of occurrence of irregular fluctuations.

Secular trend denotes continuous movement in the same direction. But, cyclical

Check your Progress

7. What are random fluctuations?

Space for hints

fluctuations do not tell us movements in the same direction; instead they tell us alterations of uptrend and down-trend in economic activity.

Seasonal variations are recurring at regular intervals of time whereas cyclical variations are recurring but not at regular intervals of time.

For trend analysis usually annual figures are used. But, for measuring seasonal and cyclical fluctuations quarterly, monthly, weekly figures are used.

Irregular fluctuations are mainly due to unpredictible events and hence its direct measurement is not possible. Usually, it is calculated as the residual component in a time series after the other three components have been accounted for.

10 Measurement of Secular Trend

10.1 Free Hand Method :

In this method the graph of the given time series data is obtained first, by taking the time along the x-axis and values of the given variable along the y-axis. Then a line is drawn through the points which in the statistician's point of view best describes the average long-term growth. There is no specific rule laid down for drawing such a line. But one practical guide is to keep the lowest points and the peaks of the graph of the given data at more or less equal distances. The line drawn is called the trend line.

We can illustrate the method of drawing the trend line by means of the following example.

Example :

In the table below we are given the sales figures for nine years. Determine the trend line by the free-hand method.

Year	Sales (in Lakhs of units)
1958	65
1959	95
1960	80
1961	115
1962	105
1963	135
1964	125
1965	150
1966	140

We mark the years from 1958 to 1966 along the x-axis and the sales figures along the y-axis.

Above the year 1953 and opposite to the sales figures 65 we plot a point.

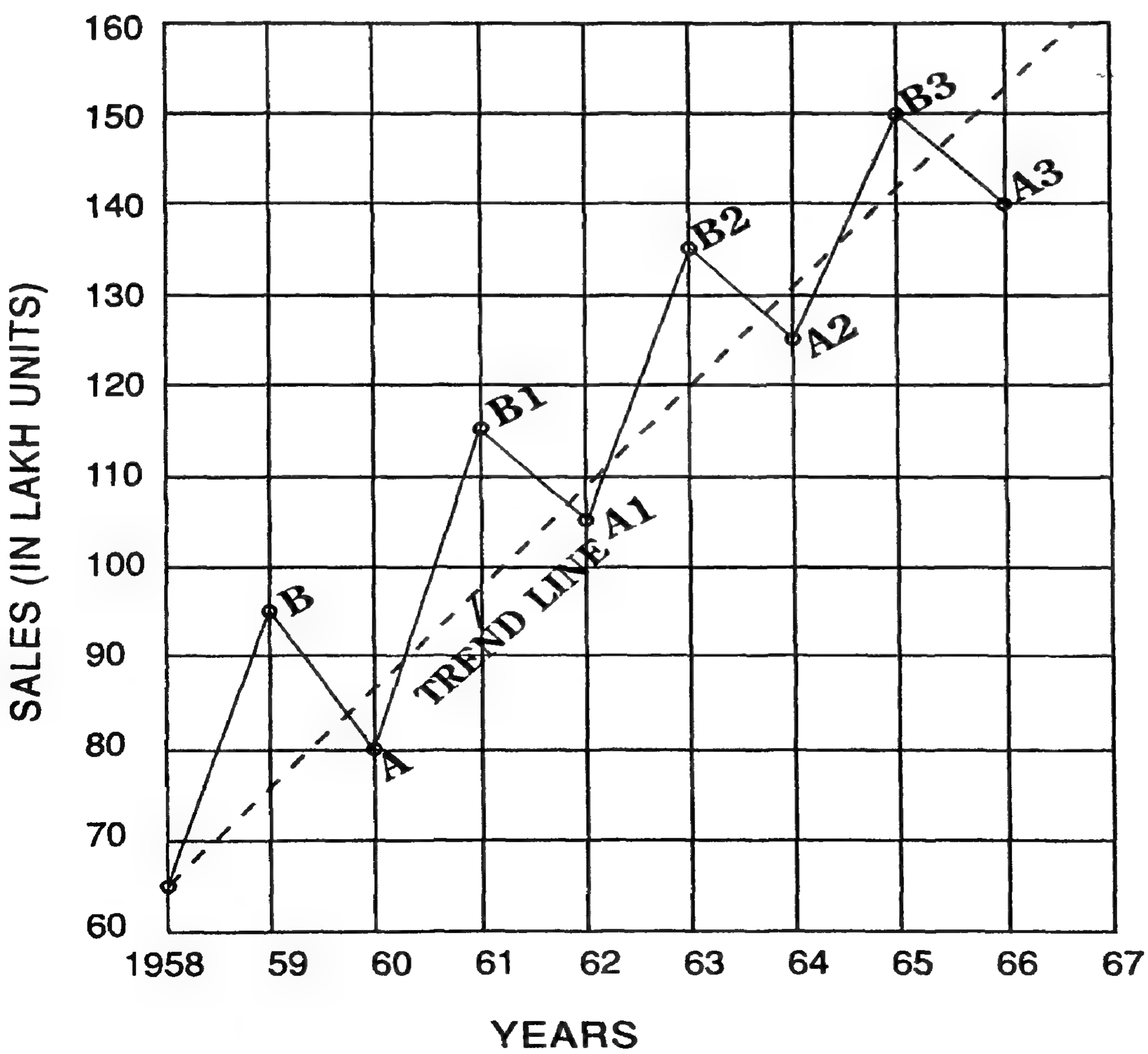
Above the year 1959 and opposite to the sales figure 95 another point is plotted.

In the same way, the sales figures for the remaining years are also marked by means of points.

All the points marked are joined by means of a straight lines. This is the graph of the given data.

We have drawn a straight-line such that the highest and the lowest points of the above graph are approximately at equal distance. That is, the vertical distances between each of the four point, B, B_1, B_2, B_3, and the straight-line should be approximately equal to the vertical distance between each of the four points A, A_1, A_2, A_3 and the straight line. This straight line is the required trend line. We have given the graph in the following figure.

TREND IN SALES



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10.1.2 Merits of the method :

i) This is the simplest method of estimating trend.

ii) Use of this method saves time; where quick results are desired this method is of great use.

iii) No mathematical complexities are involved in this method and hence this method can be easily understood.

iv) This is a quite flexible method and can be used for all types of trend, linear or non-linear.

v) This method eliminates the regular and irregular fluctuations and provides a simple line showing the basic tendency over a period of time.

vi) If a person has long experience in computing trend and has knowledge of the economic history of the concerned industry under analysis, he can fit a trend line even superior to one derived by mathematical means. Thus this method has considerable merit in the hands of experienced statisticians and is widely used in applied situations.

10.1.3 Demerits of the method :

i) This method is only an inspectional method of measuring trend. Hence it cannot be used confidently to make prediction.

ii) The trend line obtained in this method can be affected by the bias of the statistician as there are no said instructions to draw the line. Different trend lines may be obtained for the same data by different persons. Hence its application calls for sound judgement.

iii) Considerable practice is required to make a good fit and hence this method cannot be recommended for beginners.

10.2 Moving Average Method

10.2.1 Method explained

The moving average method is a simple process of trend measurement which is quite accurate under certain conditions. The meaning and method of calculation of moving average can be understood easily with the help of an example. So, let us consider the following example :

Year	Sales (in lakhs of units)
1958	65
1959	95
1960	80
1961	115
1962	105
1963	135
1964	125
1965	150
1966	140

Firstly, let us consider the sales for the first three year viz, 1958, 1959 and 1960. They are 65, 95 and 80 respectively. Arithmetic mean of three values is found out.

$$\text{Their arithmetic mean} = \frac{65 + 95 + 80}{3} = \frac{240}{3} = 80$$

Now this mean value is written opposite to the year 1959.

Secondly, we drop out the first year viz, 1958 and include the fourth year viz, 1961. That is we consider the years 1959, 1960 and 1961. Sales figures in these three years are : 95, 80 and 115 respectively. Their arithmetic mean is found out,

$$\text{Their mean} = \frac{95 + 80 + 115}{3} = \frac{290}{3} = 96.67$$

Their mean value is written opposite to the year 1960.

Thirdly, we drop out the second year viz, 1959 and include the year 1962 and find out the mean of the sales figures in the years 1960, 1961 and 1962. Their mean,

$$= \frac{80 + 115 + 105}{3} = \frac{300}{3} = 100$$

This mean value is written opposite to the year 1961.

Fourthly, the year 1960 is dropped out and the year 1963 in included. The mean of the sales figures in the three years 1961, 1962 and 1963 is,

Space for hints

Space for hints

$$= \frac{115 + 105 + 135}{3} = \frac{355}{3} = 118.33$$

This mean value is written opposite to the year 1962.

Fifthly, the year 1961 is dropped out and the year 1964 is included. The mean of the sales figures in the year 1962, 1963 and 1964 is,

$$= \frac{105 + 135 + 125}{3} = \frac{365}{3} = 121.67$$

This mean value is written opposite to the year 1963.

Sixthly, the year 1962 is dropped out and the year 1965 is included and the mean of the sales figures in 1963, 1964 and 1965 is found out. The mean,

$$= \frac{135 + 125 + 150}{3} = \frac{410}{3} = 136.67$$

This mean value is written opposite to the year 1964.

Seventhly, the year 1963 is dropped out and the year 1966 is included and the mean of the sales figures in 1964, 1965 and 1966 is found out. The mean,

$$= \frac{125 + 150 + 140}{3} = \frac{415}{3} = 138.33$$

This mean value is written opposite to the year 1965.

Now if we drop out the year 1964, we do not have a next year after 1966 to include and to get the mean value of the sales figures in the three years 1965, 1966 and 1967. So we stop at this stage.

Now we have mean values for each starting from 1959 to 1965. These mean values are called 'moving averages' for the years from 1959 to 1965. We have found out the mean values by taking the sales figures for three years in each time. Therefore these averages are called "three year moving averages".

While finding out the moving average for each of the years from 1959 to 1965 we divide the sum of the sales figures for three years by three and get the average. The sum of the sales figures of three years calculated each time is called the 'moving total'. For example, the moving total for the year 1959 is the sum of the sales figures of the years 1958, 1959 and 1960 and it is 240.

Now we give the moving totals and the moving average calculated above in a table as follows:

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Year (1)	Sales (in lakhs of units) (2)	Three year Moving totals (3)	Three year Moving averages (4)
1958	65	-----	-----
1959	95	240	80.00
1960	80	290	96.67
1961	115	300	100.00
1962	105	355	118.33
1963	135	365	121.67
1964	125	410	136.67
1965	150	415	138.33
1966	140	-----	-----

In the above table it is to be noted that there is no moving average value for the first and last years viz, 1958 and 1966.

Above we have averaged the sales figures for three years each time and hence the period of three years is called the period of the moving average.

Instead of three years, we can also take 5 or 7 years figures each time and find out the average. In such a case, the moving averages are called 5 year moving averages or 7 year moving averages and the period of the moving average is 5 years or 7 years respectively.

To calculate 5 years moving averages, first 5 years are taken first and the mean of the sales figures for these 5 years is found out and written opposite to the third year which is the middle year of the first 5 years. Nextly, the first year is dropped out and the sixth year is included and the mean of the figures for the 5 years is found out. This mean is written opposite to the fourth year. Similarly, the second year is dropped out and the seventh year is included and the average of the figures for the five years is found out, and so on. This process goes on until the last five years are considered and the average is calculated. For the data we have given earlier we can give the 5 year moving average in a table as follows.

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Year (1)	Sales (in lakhs of units) (2)	5 Year Moving totals (3)	5 Year Moving averages (4)
1958	65	-----	-----
1959	95	-----	-----
1960	80	460	92
1961	115	530	106
1962	105	560	112
1963	135	630	126
1964	125	655	131
1965	150	-----	-----
1966	140	-----	-----

In the above table it is to be noted that there is no average for the first two and the last two years.

10.2.2 Definition of moving average :

From the above examples, we can define moving averages as follows :

A moving average is an average of a fixed number of items in a time series which move through the series by dropping the top items of the previous averaged group and adding the next item below in each successive average.

Thus moving averages may be considered as an artificially constructed time series in which each periodic figure is replaced by the mean of the value of that period and those of a number of preceding and succeeding period.

10.2.3 Weighted or Centred Moving Average :

In the moving average considered in the above examples period of moving average is odd number of years like 3 years or 5 years. And we have written the average calculated opposite to the middle of the years each time.

Sometimes we may have to take even number of years like 2 or 4 or 8 years as the period of the moving average. In such a case, the average is written opposite to the gap in between the corresponding parts of middle years. For example suppose we are given the sales figures for each year from 1958 to 1966. Suppose 4 year moving averages are to be calculated. Here, the moving averages for the years 1958, 1959, 1960 and 1961, is written opposite to the gap in between the two middle years 1959 and 1960. The moving average for the years 1959, 1960, 1961 and 1962 is written opposite to the gap in between 1960 and 1961. Similarly, the remaining moving averages are also written.

Here the moving average written opposite to the gap in between 1959 and 1960 does not represent either 1959 or 1960 as it falls between December 31, 1959 and January 1, 1960. Similarly, the other moving averages also do not coincide with the given years. Hence this kind of placement of moving averages is inconvenient. In order to get over this difficulty we adjust these averages so that they may coincide with the years given. This type of adjustment is called 'centering the moving average'. The centering is done in the following manner.

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The first and second four-year moving averages are added and the total is divided by two and written opposite to the gap in between the first two four-year moving averages.

The second and third four-year moving averages are added and the total is divided by two and written opposite to the gap in between the second and third four-year moving averages and so on.

In short, two-year moving averages of the four-year moving averages are calculated. They are written opposite to the gap in between the successive parts of four-year moving averages.

The two-year moving averages of the four-year moving averages are called the 'centred four year moving averages'. These CENTRED averages will coincide with the years given.

In the following table we have calculated the CENTRED four year moving averages.

Year (1)	Sales (in lakhs of units) (2)	4 year Moving totals (3)	4 year Moving averages (4)	2 year moving total of the 4 year Moving averages (5)	CENTRED 4 year Moving averages (6)
1958	65	-----	-----	-----	------
1959	95	355	88.75	-----	-----
1960	80	395	98.75	187.50	93.75
1961	115	435	108.75	207.50	103.75
1962	105	480	120.00	228.75	114.38
1963	135	515	128.75	248.75	124.38
1964	125	550	137.5	266.25	133.13
1965	150	-----	-----	-----	------
1966	140	-----	-----		

Space for hints

There is no CENTRED average for the first two and the last two years.

If we look at the table given above showing 4 year CENTRED moving averages, we can note that the procedures indicated in that table are more laborious than necessary. We need not calculate the moving averages in column (4). Instead by computing a two year moving total of the figures in column (3) and dividing each of these totals by 8, we can get exactly the same figures in column (6).

For example, the CENTRED 4-year moving averages for 1960 can be obtained by adding the first two 4 year moving totals in column (3), namely, 355 and 395 and dividing the sum by 8. That is, $\frac{355+395}{8} = \frac{750}{8} = 93.75$. Similarly, the CENTRED 4-year moving average for 1961 is obtained by adding the two moving totals 395 and 435 and dividing the sum by 8. That is, $\frac{395+435}{8} = \frac{830}{8} = 103.75$.

The CENTRED 4-year moving average for 1962 is, $\frac{435+480}{8} = \frac{915}{8} = 114.38$.

The CENTRED 4-year moving average for 1963 is, $\frac{480+515}{8} = \frac{995}{8} = 124.38$.

The CENTRED 4-year moving average for 1964 is, $\frac{515+550}{8} = \frac{1065}{8} = 133.13$.

This procedure reduces the calculation of 4 year CENTRED moving averages to some extent. However, there is an even more expeditious procedure and it is as follows :

Consider the 4-year CENTRED moving average for the year 1960. It is obtained by adding the two moving totals 355 and 395 and dividing the sum by 8. The moving total 355 is obtained by adding the values for the first four years namely, 1958, 1959, 1960 and 1961 (i.e.) 355 = 65 + 95 + 80 + 115. The moving total 395 is obtained by adding the values for the second four years, namely 1959, 1960, 1961 and 1962 (i.e.) 395 = 95 + 80 + 115 + 105. Therefore, 355 + 395 = (65 + 95 + 80 + 115) + (95 + 80 + 115 + 105) = 1 x 65 + 2 x 95 + 2 x 80 + 2 x 115 + 1 x 105 = Sum of the value for 1958, twice the value for 1959, twice the value for 1960, twice the values for 1961, and the value for 1962) = The weighted total of the values for the five years 1958 to 1962 the weights for the years being 1,2,2,2,1 respectively.

Sum of these weights = 1 + 2 + 2 + 2 + 1

= 8

∴ 4-year CENTRED moving average for the year 1960

$$= \frac{\text{Weighted total of the values for the years from 1958 to 1962 with the weights 1,2,2,2,1}}{\text{Sum of the weights 1,2,2,2,1}}$$

That is, the 4-year CENTRED moving average for the year 1960 is simply the weighted arithmetic mean of the values for the five years from 1958 to 1962, with

the weights 1,2,2,2,1 respectively. This weighted arithmetic mean is called the weighted five years, moving average with the weights 1,2,2,2,1 respectively.

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In a similar manner we can show that the 4-year CENTRED moving average for the year 1961 is the weighted arithmetic mean of the values for the five years 1959 to 1963 with the weights 1,2,2,2,1 respectively.

$$= \frac{1 \times 95 + 2 \times 80 + 2 \times 115 + 2 \times 105 + 1 \times 135}{1 + 2 + 2 + 2 + 1}$$

$$= \frac{95 + 160 + 230 + 210 + 135}{8}$$

$$= \frac{830}{8}$$

$$= 103.75$$

Thus, the 4-year CENTRED moving average for the year 1961 is the same as the weighted 5 year moving average of the five years 1959 to 1963 with the weights 1,2,2,2,1 respectively.

Similarly, we can show for the remaining years also.

Therefore, in general, the CENTRED 4 year moving averages are the same as the weighted 5 year moving averages with the weights 1,2,2,2,1 respectively.

In the following table we have given the weighted 5 years moving averages with the weights 1,2,2,2,1 respectively.

Year (1)	Sales (in lakh units) (2)	Weighted 5-year moving totals (3)	Weighted 5-year moving averages (4)
1958	65	—	—
1959	95	—	—
1960	80	$1\times65+2\times95+2\times80+2\times115+1\times105=$ **750**	750/8 = 93.75
1961	115	$1\times95+2\times80+2\times115+2\times105+1\times135=$ **830**	830/8 = 103.75
1962	105	$1\times80+2\times115+2\times105+2\times135+1\times125=$ **915**	915/8 = 114.38
1963	135	$1\times115+2\times105+2\times135+2\times125+1\times150=$ **995**	995/8 = 124.38
1964	125	$1\times105+2\times135+2\times125+2\times150+1\times140=$ **1065**	1065/8 = 133.13
1965	150	—	—
1966	140	—	—

Space for hints

It is to be noted that the figures in column (4) of the above table (showing 5-years weighted moving averages) are the same as those in column (6) of the previous table (showing 4 year CENTRED moving averages).

Instead of 4-year CENTRED moving averages, if we want 2-year CENTRED moving averages, they can be obtained by finding out 3-year weighted moving averages with the weights 1,2,1 respectively.

6-year CENTRED moving averages can be obtained by finding out 7-year weighted averages with the weights 1,2,2,2,2,2,1 respectively.

8-year CENTRED moving averages can be obtained by finding out 9-year weighted moving averages with the weights 1,2,2,2,2,2,2,2,1 respectively and so on.

From above we can define weighted moving averages as follows :

A weighted moving average is simply a CENTRED moving averages and the period of the weighted moving average is always greater than that of the CENTRED moving average by one year. Each time when a weighted moving average is calculated, the two extreme years are given the weights 1,1 and each of the middle year is given the weight 2.

Example-3 :

Find the trend for the following series by three year weighted moving average with weights 1,2,1.

Year	1	2	3	4	5	6	7
Production series	2	4	5	7	8	10	13

In the following table we have calculated the three year weighted moving averages.

Year (1)	Production Series (2)	3-year weighted moving totals (3)	3-year weighted moving averages (4)
1	2	-----	-----
2	4	(1 x 2) + (2 x 4) + (1 x 5) = 15	15/4 = 3.75
3	5	(1 x 4) + (2 x 5) + (1 x 7) = 21	21/4 = 5.25
4	7	(1 x 5) + (2 x 7) + (1 x 8) = 27	27/4 = 6.75
5	8	(1 x 7) + (2 x 8) + (1 x 10) = 33	33/4 = 8.25
6	10	(1 x 8) + (2 x 10) + (1 x 13) = 41	41/4 = 10.25
7	13	-----	-----

Space for hints

In the examples we have considered so far, the time series data were of yearly data. But, the data can consist of monthly or daily figures also. In such cases also the moving average is calculated in the same manner. When the data are monthly figures, 2 months moving average 3 months moving average or 5 months moving average etc., can be calculated. When the data are weekly figures 2 weeks or 4 weeks moving average are calculated. When the data are daily figures moving averages of 2 days, 3 days or 4 days are calculated.

The selection of the period for calculating average is the crux of this method. The main purpose of this method is to obtain the trend values so that all types of fluctuations are eliminated or in any case reduced to a minimum. The period of moving average would be such as to achieve these objectives.

A long period moving average is good from the point of view of reducing irregular fluctuations. But such a moving average is likely to distort trend values. Longer the period of the moving average the farther away are the trend values from the original one. Hence, the period of the moving average should neither be too long to distort the trend values nor be too short to make it impossible to reduce irregular fluctuations.

Whenever there is cyclical variation in the given series, the moving average method can be employed advantageously. This method is best suited for the data which are characterised, by periodic movements. This method fully wipes off the periodic fluctuations, reduces irregular fluctuations and gives us the best possible trend values if the period selected for the moving average coincides with the period of the cycle.

It is likely that the period of the cycle in a series is not uniform. In such a situation, there is the problem of selecting the proper period for calculating moving averages. It is suggested that in such cases we should take moving average period equal to or some what greater than the average period of the cycle in the data.

As the moving average values represent the trend value the difference between the values in the original series and the moving averages give us the short term fluctuations.

Example :

Find the trend of bank clearance by the method of moving average taking 7 years as the period of moving average. Also find out the values of short-term oscillations.

Space for hints

Year	Bank clearance (Rs. Crores)
1916	53
1917	79
1918	76
1919	66
1920	69
1921	94
1922	105
1923	87
1924	79
1925	104

To calculate the 7 year moving average first we consider the figures for the first 7 years viz., from 1916 to 1922. They are 53, 79, 76, 66, 69, 94 and 105 respectively. Their total is $53 + 79 + 76 + 66 + 69 + 94 + 105 = 542$. Their mean $\frac{572}{7} = 77.43$. We write the mean value against the year 1919 which is the middle year of the 7 year considered.

Next we drop the first year viz., 1916 and include the year viz, 1923. The mean of the figures for the years starting from 1917 to 1923 is $\frac{79 + 76 + 66 + 69 + 94 + 105 + 87}{7} = \frac{576}{7} = 82.3$. We write this value opposite to the middle of the years from 1917 to 1923 (i.e.,) opposite to the year 1920. Next we drop out the year 1917 and include the year 1924. Now the mean of figures for the 7 years from 1918 to 1924 is $\frac{76 + 66 + 69 + 94 + 105 + 87 + 79}{7} = \frac{576}{7} = 82.3$.

This average is written opposite to the middle year 1921.

Next we drop out the year 1918 and include the year 1925. These are the last seven years. Mean of the figures for these last seven years $\frac{66 + 69 + 94 + 105 + 87 + 79 + 105}{7} = \frac{604}{7} = 86.3$.

This mean value is written opposite to the middle year of the last 7 years viz, 1922.

Now we give the moving averages in a table as follows :-*

Year (1)	Bank clearance (Rs. crores) (2)	7 year moving totals (3)	7 year moving averages (4)
1916	53	-----	-----
1917	79	-----	-----
1918	76	-----	-----
1919	66	542	77.43
1920	69	576	82.30
1921	94	576	82.30
1922	105	604	86.30
1923	87	-----	-----
1924	79	-----	-----
1925	104	-----	-----

We are asked to find out the short-term oscillations also. The difference between the value in the given series and the moving average gives the value of short-term oscillation. Therefore we find out the short-term oscillation as follows :

For the year 1919, value in the given series is 66 and the value of moving average is 77.43

$\therefore$ Short-term oscillation for 1919 = 66 – 77.43

$= -11.43$

For the year 1920, value in the given series is 69 and the value of moving average is 82.3

$\therefore$ Short-term oscillation for 1920 = 69 – 82.3

$= -13.3$

For the year 1921, value in the given series is 94 and the value of moving average is 82.3

$\therefore$ Short-term oscillation for 1921 = 94 – 82.3

$= 11.7$

For the year 1922, value in the given series is 105 and the value of moving average is 86.3

$\therefore$ Short-term oscillation for 1922 = 105 – 86.3

$= 18.7$

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* Just for your understanding purpose we have given all the detailed steps before giving the table. When you are asked to find out the moving average you need not reproduce these steps. It is enough if you form the table directly and give it alone.

Space for hints

Space for hints

Example :

The following data give the index numbers of export prices in India. Smooth the data by fitting a linear trend by the method of moving averages, taking 5 yearly periods.

Year	Index No.	Year	Index No.
1938 - 39	100.0	1946 - 47	284.9
1939 - 40	119.8	1947 - 48	372.2
1940 - 41	130.3	1948 - 49	421.4
1941 - 42	155.9	1949 - 50	435.7
1942 - 43	184.6	1950 - 51	482.9
1943 - 44	227.4	1951 - 52	711.7
1944 - 45	244.2	1952 - 53	500.0
1945 - 46	240.8	1953 - 54	461.0

We have calculated the 5 yearly moving averages and given in following table :

Year (1)	Index No. (2)	5 yearly moving total (3)	5 yearly moving averages (4)
1938 - 39	100.0	-----	-----
1939 - 40	119.8	-----	-----
1940 - 41	130.3	690.6	138.12
1941 - 42	155.9	818.0	163.60
1942 - 43	184.6	942.5	188.50
1943 - 44	227.4	1053.0	210.60
1944 - 45	244.2	1182.2	236.44
1945 - 46	240.8	1369.6	273.92
1946 - 47	284.9	1563.6	312.72
1947 - 48	372.2	1755.0	351.00
1948 - 49	421.4	1997.1	399.42
1949 - 50	435.7	2423.9	484.78
1950 - 51	482.9	2551.7	510.34
1951 - 52	711.7	2591.3	518.26
1952 - 53	500.0	-----	-----
1953 - 54	461.0	-----	-----

Example :

Compute the trend values of the following temperature readings in degrees of Fahrenheit by the method of moving averages. (Take the period of moving average to be 5 days).

Date	Temperature
1991 Feb. 1	40
,, 2	50
,, 3	44
,, 4	70
,, 5	52
,, 6	44
,, 7	36
,, 8	40

We have calculated the 5 days moving averages and given in the following table :

Date (1)	Temperature (2)	5 day Moving total (3)	5 day Moving average (4)
1991 Feb. 1	40	-----	-----
,, 2	50	-----	-----
,, 3	44	256	51.2
,, 4	70	260	52.0
,, 5	52	246	49.2
,, 6	44	242	48.4
,, 7	36	-----	-----
,, 8	40	-----	-----

Example :

Calculate the trend values by 5 yearly moving average, plot the original data and trend on the same graph paper.

Years	Value	Years	Value
1950	105	1956	85
1951	115	1957	75
1952	100	1958	60
1953	90	1959	65
1954	80	1960	70
1955	95	1961	55

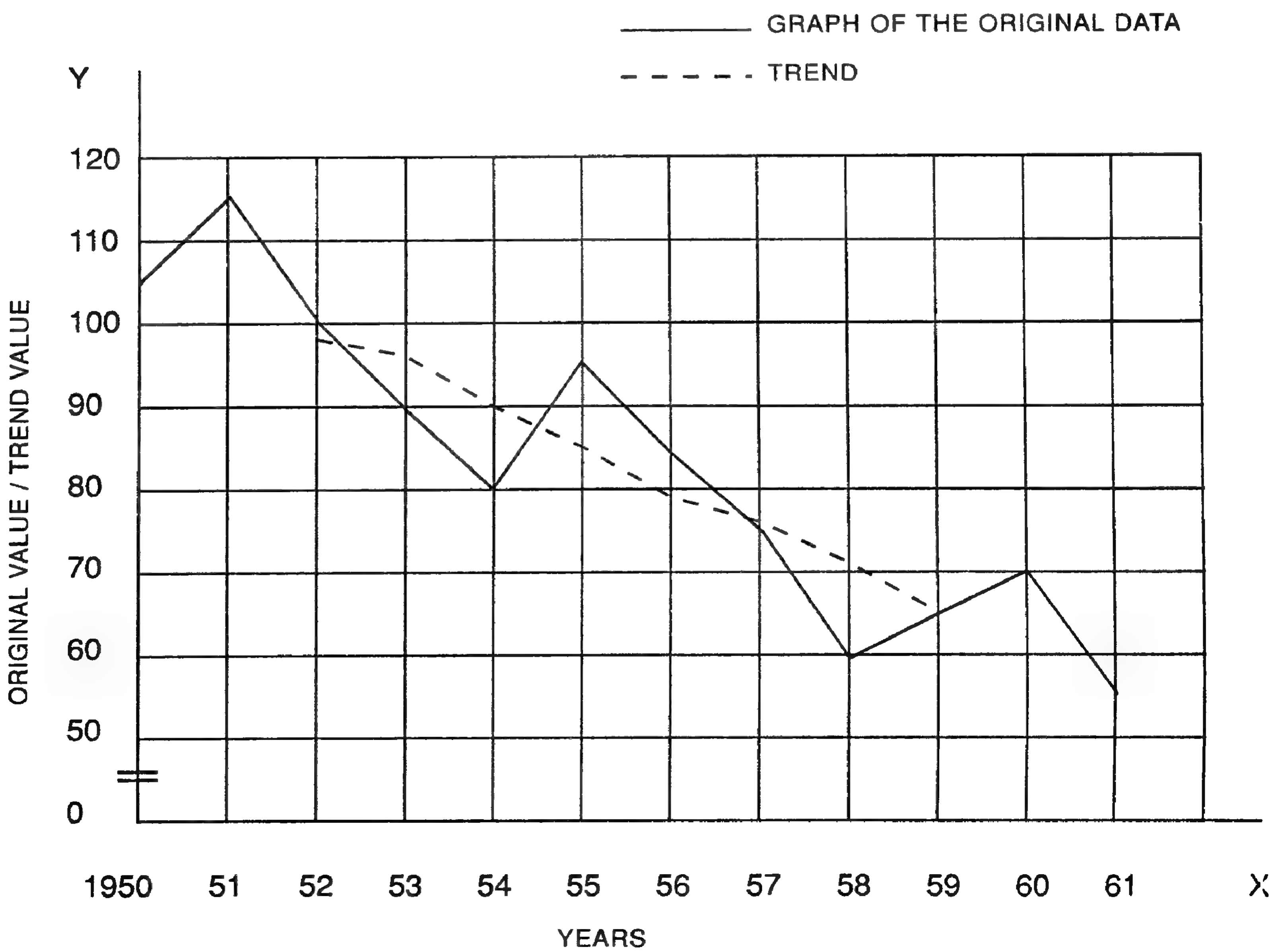
Space for hints

Space for hints

We calculate the trend values by 5 yearly moving average as follows :

Year (1)	Value (2)	5 yearly moving totals (3)	5 yearly moving averages (4)
1950	105	-----	-----
1951	115	-----	-----
1952	100	490	98
1953	90	480	96
1954	80	450	90
1955	95	425	85
1956	85	395	79
1957	75	380	76
1958	60	355	71
1959	65	325	65
1960	70	-----	-----
1961	55	-----	-----

Graph of the original data and the trend are as follows:



10,2,4 Merits of moving average method.

Space for hints

i) Moving average method is a simple device of reducing fluctuations and obtaining trend values with a fair degree of accuracy.

ii) As the free-hand method this method is not subject to personal prejudice and bias of the estimator.

iii) When the period of moving average is taken as the period of the cycle, the cyclical variations are completely eliminated.

iv) This method is a flexible method of estimating trend. For, the inclusion of few more values to the data does not change the entire calculation but it yields some more trend values.

10.2.5 Demerits of moving average method.

i) The choice of the period of moving average needs a great amount of care. If an inappropriate period is selected, a true picture of the trend cannot be obtained.

ii) If the series given is a very large one, then the calculation of moving average is cumbersome.

iii) The moving average is calculated with the help of arithmetic mean. Hence, it is very much affected by the extreme values. That is, it is affected by a few very high or low values to a considerable extent.

iv) The moving average technique does not give trend value at each end and hence it cannot be brought upto date. One of the most important services of time series trend is its use in forecasting future change. This will not be possible unless the trend is brought upto date.

v) The moving average should not be used for very short time series.

vi) Theoretically speaking it is true that if the period of the moving average coincides with the period of the cycle, the cyclical variation are completely eliminated. But, in practice cycles are by no means perfectly periodic and the length of various cycles in any given series usually vary considerably, and therefore, no moving average can completely remove the cycle. The best results would be obtained by a moving average whose period is equal to the average length of all the cycles in the given series. However, it is difficult to determine the average length of the cycles until the cycles are isolated from the series.

vii) If the moving averages are calculated for a series of complete random variations, they tend to produce a series with spuriously periodic elements in it.

Space for hints

10.3 Method of Least Squares

10.3.1 Method Explained

The method of least squares is a widely used method of fitting a curve for the given data. It is the most popular method used to determine the position of the trend line of a given time series. In this method a mathematical relationship is established between the time factor and the variable given. The procedure to fit a linear or straight line trend by this method is explained as follows :

Suppose the given time series data represent the value of a variable for n years. For example if the given time series shows the sales figures for 10 years starting from 1958 to 1967 then n = 10.

We represent the given values of the variable by the letter y. All the values of y are summed up and the sum is denoted by Σy.

We take the middle year of given period as origin. If the given period consists of an odd number of years, we will have only one middle year. For example, if we are given the value of a variable for nine years starting from 1958 to 1966 then the fifth year viz. 1962 will be the middle year. On the other hand, if the given period consists of even number of years, we will have two middle years. In such cases we find out the mean of the two middle years and take it as the origin. For example, suppose we are given the values of a variable for 10 years successively starting from 1958 to 1967. Here, the years 1962 and 1963 will be the middle years. Mean of these two years is 1962.5. This 1962.5 is taken as the origin.

Now we find out the deviation of each year from the middle year viz, origin. We denote these deviations by the letter x.

Square of each value of x (i.e.) the values of x^2 are found out and summed up. The sum is denoted by Σx^2.

Each value of x is multiplied by the corresponding value of y. That is, the values of xy are found out. These values are also summed up and denoted by Sxy.

Now, the equation to the trend line is given by the formula

$$y = a + bx$$

where $a = \dfrac{\Sigma y}{n}$

$$b = \frac{\Sigma xy}{\Sigma x^2}$$

Check your Progress

8. Name the different methods of measuring trend.

Now putting each value of x in the equation y = a + bx, the value of y is obtained and it is the trend value. Thus the trend values of given time series are obtained.

Space for hints

Sometimes the equation to the trend line is given in the following form also ;

$y = mx + C$

Here, instead of a, we have C and instead of b, we have m

K $C = \frac{\Sigma y}{n}$ and $m = \frac{\Sigma xy}{\Sigma x^2}$

Now, we can find the values of C and m and get the equation to the trend line.

The method of finding out the trend values by this method can be illustrated by means of the following examples :

Example-1 :

Fit a straight line trend by the method of least squares from the following data and find the trend values.

Year	Sale in lakh units
1958	65
1959	95
1960	80
1961	115
1962	105

Here we are given the sales figure (in lakh units) for 5 years

$\therefore n = 5.$

We represent the sales figures given by the letter y. Hence the sales figures are given under the heading y in the table below.

Since we are given a period of 5 years which is an odd number of year we have only one middle year. The middle year is the third year given viz, 1960. We take 1960 as the origin.

Now the deviation of each year from 1960 is found out

Deviation of the year 1958 from 1960 = 1958 – 1960 = –2

Deviation of the year 1959 from 1960 = 1959 – 1960 = –1

Deviation of the year 1960 from 1960 = 1960 – 1960 = 0

Deviation of the year 1961 from 1960 = 1961 – 1960 = 1

Deviation of the year 1962 from 1960 = 1962 – 1960 = 2

Space for hints

These deviations are the values of x. Therefore, these deviations viz., -2, -1, 0, 1, 2 are given under the heading x in the table below.

Squares of the values of x are $(-2)^2$, $(-1)^2$, $(0)^2$, $(1)^2$ and $(2)^2$. That is, 4, 1, 0, 1 and 4 are the squares of the values of x. That is, they are the values of x^2 and hence, they are given under the heading x^2 in the table below.

Now each value of x is multiplied by the corresponding value of y. That is, the values of xy are found out. The values of xy are, [(–2) x 65] [(–1) x 95] [0 x 80] [1 x 115] and [2 x 105]. That is, the values of xy are –130, –95, 0, 115 and 210. These are given under the heading xy in the table below.

Now we give the table as follows :

Year	y	x	x^2	xy
1958	65	–2	4	–130
1959	95	–1	1	–95
1960	80	0	0	0
1961	115	1	1	115
1962	105	2	4	210
Total	460	---	10	100

$\Sigma y = 460$

$\Sigma x^2 = 10$

$\Sigma xy = 100$

$n = 5$

Equation to the straight line trend is

$$y = a + bx \text{ where } a = \frac{\Sigma y}{n} \text{ and } b = \frac{\Sigma xy}{\Sigma x^2}$$

$$\therefore \quad a = \frac{\Sigma y}{n}$$

$$= \frac{460}{5}$$

$$= 92$$

$$b = \frac{\Sigma xy}{\Sigma x^2}$$

Space for hints

$$= \frac{100}{10}$$

$$= 10$$

$\therefore$ The equation to the straight line trend is $y = 92 + 10x$. The trend values are obtained as follows:

For the year 1958, $x = -2$.

From the equation we have derived, the value of y when $x = -2$ is

$y = 92 + 10(-2)$

$= 92 - 20$

$= 72$

72 is the trend value for the year 1958.

Similarly, for the year 1959 $n = -1$, and hence the trend value,

$y = 92 + 10(-1)$

$= 92 - 10$

$= 82$

For the year 1960, $x = 0$ and hence the trend value

$y = 92 + (10 \times 0)$

$= 92 + 0$

$= 92$

For the year 1961, $x = 1$ and hence the trend value

$y = 92 + (10 \times 1)$

$= 92 + 10$

$= 102$

For the year 1962, $x = 2$ and hence the ı. end value

$y = 92 + (10 \times 2)$

$= 92 + 20$

$= 112$

Space for hints

Example-2 :

Find out the straight line trend and trend values by the method of least squares from the following data. Also find out the expected value for the year 1938.

Year	No. of industrial failures
1929	23
1930	26
1931	28
1932	32
1933	20
1934	12
1935	12

We are given the values of the variable viz, number of industrial failures, for seven years.

$\therefore$ n = 7

We denote the number of industrial failures by the letter y and the numbers are given under the heading y in the table below :

Since an odd number of years is given, we have only one middle year. The middle year is 1932. Therefore, 1932 is the origin.

Now as in the previous example, values of x,x^2 and xy are found out and given in a tabular form as follows :

Origin = 1932

Year	y	x	x^2	xy
1929	23	–3	9	(–3) x 23 = –69
1930	26	–2	4	(–2) x 26 = –52
1931	28	–1	1	(–1) x 28 = –28
1932	32	0	0	0 x 32 = 0
1933	20	1	1	1 x 20 = 20
1934	12	2	4	2 x 12 = 24
1935	12	3	9	3 x 12 = 36
Total	153		28	–69

Space for hints

$\Sigma y = 153$

$\Sigma x^2 = 28$

$\Sigma xy = -69$

$n = 7$

The equation to the straight line trend is

$y = a + bx$ where $a = \dfrac{\Sigma y}{n}$; and $b = \dfrac{\Sigma xy}{\Sigma x^2}$

$$\therefore \quad a = \frac{\Sigma y}{n}$$

$$= \frac{153}{7}$$

$$= 21.857$$

$$b = \frac{\Sigma xy}{\Sigma x2}$$

$$= \frac{-69}{28}$$

$$= -2.46$$

$\therefore$ The equation to the straight line trend is

$y = 21.857 - 2.46x$

The trend values are found out as follows :

For 1929, $x = -3$ $\quad \therefore y = 21.857 - 2.46(-3)$

$= 21.857 + 7.38$

$= 29.237$

For 1930, $x = -2$ $\quad \therefore y = 21.857 - 2.46(-2)$

$= 21.857 + 4.92$

$= 26.777$

For 1931, $x = -1$ $\quad \therefore y = 21.857 - 2.46(-1)$

$= 21.857 + 2.46$

$= 24.317$

For 1932, $x = 0$ $\quad \therefore y = 21.857 - 2.46 \times 0$

$= 21.857$

357 – 2.46 x 1

57 – 2.46

397

For 1934, x = 2 $\quad\therefore$ y = 21.857 – 2.46 x 2

= 21.857 – 4.92

= 16.937

For 1935, x = 3 $\quad\therefore$ y = 21.857 – 2.46 x 3

= 21.857 – 7.38

= 14.477

We are asked to find out the expected value for the year 1938. For this we find out the value of x for this year 1938.

For 1938 value of x = 1938 – 1932 = 6

We put this value of x in the equation we have derived and get the value of y and it is the required value.

When x = 6, y = 21.857 – 2.46 x 6

= 21.857 – 14.76

= 7.097

7.097 is the expected value, for the year 1938.

Example-3 :

Fit a straight line of the form y = mx + C for the following time series showing production of a commodity over a period of eight years.

Year	1961	1962	1963	1964	1965	1966	1967	1968
Production in lakhs	8	12	15	18	20	23	27	30

Let y represent production in lakhs

n = number of years given

$\therefore$ 4th as well as the 5th years are the middle years (i.e. 1964 and 1965 are the middle years).

∴ To find out the deviation of each year we take 1964.5 as the origin and get the values of x.

Year	y	x	x^2	xy
1961	8	–3.5	12.25	–28.0
1962	12	–2.5	6.25	–30.0
1963	15	–1.5	2.25	–22.5
1964	18	–.5	.25	–9.0
1965	20	.5	.25	10.0
1966	23	1.5	2.25	34.5
1967	27	2.5	6.25	67.5
1968	30	3.5	12.25	105.0
Total	153		42.00	127.5

$\Sigma y = 153$

$\Sigma x^2 = 42$

$\Sigma xy = 127.5$

$n = 8$

$$K = m = \frac{\Sigma xy}{\Sigma x^2} = \frac{127.5}{42} = 3.03$$

$$C = \frac{\Sigma y}{n} = \frac{153}{8} = 19.125$$

∴ Equation to the straight line trend is

$y = 3.03\, x + 19.125$

Example-4 :

Find out the trend of industrial employment in a city from the following data from 1955 to 1964 by the method of least square.

Space for hints

Year	No. of workers Employed (lakhs)
1955	3.2
1956	3.3
1957	3.7
1958	3.9
1959	3.6
1960	4.3
1961	4.8
1962	4.8
1963	5.4
1964	6.3

We are given values for 10 years

$\therefore$ n = 10

Since we are given a period consisting of even number of years, there are two middle years. The two middle years are 1959 and 1960. Therefore, the mean of the two years 1959 and 1960 viz, 1959.5 is taken as the origin.

Having chosen the origin the remaining procedure to find out the trend line is the same as in the previous cases. Therefore now we form the table as follows :

Origin = 1959.5

Year	y	x	x^2	xy
1955	3.2	–4.5	20.25	–14.40
1956	3.3	–3.5	12.25	–11.55
1957	3.7	–2.5	6.25	–9.25
1958	3.9	–1.5	2.25	–5.85
1959	3.6	–0.5	.25	–1.80
1960	4.3	0.5	.25	2.15
1961	4.8	1.5	2.25	7.20
1962	4.8	2.5	6.25	12.00
1963	5.4	3.5	12.25	18.90
1964	6.3	4.5	20.25	28.35
Total	43.3		82.50	25.75

Space for hints

$\Sigma y = 43.3$

$\Sigma x^2 = 82.50$

$\Sigma xy = 25.75$

$n = 10$

$$a = \frac{\Sigma y}{n}$$

$$= \frac{43.3}{10}$$

$$= 4.33$$

$$b = \frac{\Sigma xy}{\Sigma x^2}$$

$$= \frac{25.75}{82.50} = .31$$

$\therefore$ The equation to the trend line is $y = 4.33 + .31x$

The trend values are obtained as follows :

Year	x	Trend = 4.33 + .31 x
1955	–4.5	4.33 + .31 (–4.5) = 2.935
1956	–3.5	4.33 + .31 (–3.5) = 3.245
1957	–2.5	4.33 + .31 (–2.5) = 3.555
1958	–1.5	4.33 + .31 (–1.5) = 3.865
1959	–0.5	4.33 + .31 (–0.5) = 4.175
1960	0.5	4.33 + .31 (0.5) = 4.485
1961	1.5	4.33 + .31 (1.5) = 4.795
1962	2.5	4.33 + .31 (2.5) = 5.105
1963	3.5	4.33 + .31 (3.5) = 5.415
1964	4.5	4.33 + .31 (4.5) = 5.725

Note : It is to be noted that the trend equation need not be used each time to get the trend value of each of the given year. It is enough if we use the trend equation to get

the trend value of the first year alone. By adding the value of b (or m) to this (i.e.,) to the trend value of the (first year), the trend value of the second year can be obtained. By adding the value of b to the trend value of the second year, the trend value of the third year can be obtained. By adding the value of b to the trend value of the third year, the trend value of the fourth year can be obtained and so on. This you can verify by yourself. In the above table, if you use this method in the examination, you can save time.

10.3.2 Merits :

(i) Unlike the method of moving averages, this method gives the trend values for all the years.

(ii) This method is useful to get the trend values even for a year which is not in the time series given.

(iii) This method gives the best estimates of trend values because (a) the sum of the deviation of the actual values given from the trend values obtained is zero (b) the sum of the squared deviations of the actual values is the least.

(iv) This method is highly impersonal and is completely free from personal bias. Also, the value obtained are definite. Therefore, this is the most objective method.

10.3.3 Demerits :

(i) Since this method is highly mathematical, it is difficult for the ordinary layman to understand it.

(ii) Like moving average method, this method is not flexible. If certain new values are included to the given time series fresh trend values are to be calculated because the values of n, y, x, x^2 and xy would all change.

(iii) If some values deviate from the trend by an extremely large amount, this method will tend to give too much weight to these deviations. This will give rise to a trend which does not follow the general course of the series. Time series are subject to extreme fluctuations. Therefore, this method cannot be used in every situation.

11. Comparison between the moving average method and the method of least squares :

Moving Average method	Method of Least Squares
1) We presume that there is no law on which the changes are based.	1) It is presumed that there is a definite law which governs the changes.
2) Trend values are obtained from equation the data themselves.	2) Trend values depend on the derived.
3) This method is more flexible.	3) This method is not so flexible.

12. Answers to the Check Your Progress Questions

1.	Refer	1	5.	Refer	[illegible]
2.	Refer	3	6.	Refer	7.1
3.	Refer	5.1	7.	Refer	8.1
4.	Refer	6.1	8.	Refer	10.1 to 10.3

13. Model questions for guidance:

10 Marks Questions (One Page Answer) :

1) Give an account of the method of least squares in fitting a straight line trend to a time series.

2) Explain with suitable illustration the terms :

a) Secular Trend

b) Seasonal variations, with respect to a time series

3) Explain how a linear trend can be fitted by the method or least squares.

4) Distinguish between Seasonal and Random Variation in time series.

20 Marks Questions (Three Page Answer) :

1) Detail different methods of trend elimination from observed time series indicating the types of series for which each method is appropriate.

2) Fit a straight line trend to the following data on the domestic demand for motor fuel.

Year	1946	1947	1948	1949	1950	1951	1952	1953	1954	1955	1956
Average monthly (demand million barrels)	61	66	72	76	82	90	69	100	103	110	114

Space for hints

Space for hints

3) Show how you would determine the secular trend of a time series by the method of moving averages.

4) What are the important components of a Time series? Explain each with illustrations.

5) Explain the concept of 'moving average'. How do they help to measure the secular trend in a time series?

5) Discuss the value of different methods of measuring trend in time series.

6) Obtain the trend values by fitting a straight line trend equation.

Years	1950	1951	1952	1953	1954	1955	1956
Values :	300	709	600	800	900	700	1000

7) a) Explain any one method of obtaining long-term trend of a time series.

b) What are the characteristics of 'Moving Averages'?

UNIT – 8

FUNCTIONS AND EQUATIONS

Space for hints

Introduction

Equation is an important mathematical concept without which no mathematical analysis is possible. Equation is a statement of equality of two quantities consisting of some unknowns called 'variables'. Finding the values of such variables is called solving equations. How to solve a give Equation? The procedure is explained in this Unit-8.

'Function' is a simple mathematical concept used to represent the relationship existing among two or more variables. Depending upon the number of variables and form of relationship existing among the variables, functions are classified into various types. Functions which find important applications in Economics are also explained in this Unit-8.

Unit Objectives :

After studying this Unit, you would be able to understand

* the meaning and important types of equations
* how to solve quadratic equation in two variable
* how to solve simultaneous linear equations in two variables
* the meaning and types of functions
* the equation to a straight line.

Unit Structure :

1. Some Basic Concepts
2. Equation
3. SolvingEquations
4. Functions
5. Functions and Curves
6. Equation of a straight line
7. Answers to Check Your Progress Questions
8. Model questions for guidance.

1. Some Basic Concepts

1.1 Algebraic Statement - Meaning

Consider the following statements: $2+3 = 5$; $8+4 = 12$; $5+9 = 14$: Statements such as these expressed in terms of specific numbers are called arithmetical statements. These arithmetical statements may be given in a

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generalised form as a+b = c which means that sum of any two numbers denoted by a and b is a third number denoted by c. The generalised statement made in terms of letters of alphabet is called an 'algebraic statement'. Instead of the letters of alphabet, some other symbols may also be used in algebraic statements. The letters of alphabet or symbol used in algebraic statements are known as 'literal' or 'general' or 'variable' numbers.

1.2 Shortcuts used in Algebraic Statements

In the algebraic statements, some shortcuts are used in writing literal numbers. These are explained below.

(i) In writing the product of a specific and a literal number such as $5 \times a$, the multiplication sign is ignored. That is, $5 \times a$ is written as simply $5a$.

$(-6) \times a$ is written as $-6a$.

$1 \times a$ is written as a.

(ii) The product of two different literal numbers say, $a \times b$ is written as ab.

Note : In arithmetic, 45 Stands for the number 'forty five' and not for 4×5.

(iii) In the case of multiplication of a literal number by the same number, the rule of exponents is used.

$\therefore a \times a \times a = a^3$; $b \times b \times b \times \times = b^4$

$5 \times a \times a \times a \times b \times b \times b \times b = 5a^3b^4$

Note : Exponent of a is 1.

(iv) In the case of division, in place of division symbol (÷), we frequently use the horizontal bar as in a fraction. For exampe, $5 \div a$ is written as $\frac{5}{a}$

$a \div 5$ is written as $\frac{a}{5}$; $a \div b$ is written as $\frac{a}{b}$

1.3 Algebraic Expression :

Any group of symbols properly connected by signs of operation (that is, signs of addition, subtraction multiplication and division) is known as an 'algebraic expression'.

For example, $(5a+b)$ is an algebraic expression. This expression means $(5 \times a)+b$

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1. What is meant by algebraic statement?

(a+b) (c–d) is an algebraic expression which means the product of two numbers namely, the sum of two numbers a and b and the balance obtained when d is subtracted from c.

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Similarly $\frac{(a+b)}{(c+d)}$, is an algebraic expression.

Some more examples of algebraic expressions are:

(i) $5x + 3y + 4z$ (ii) $3a^2 + 4ab^2$

(iii) $\frac{2x+3y}{4x-y}$ (iv) $9a - 5b$.

1.4 Term of an Algebraic Expression :

In an algebraic expression consisting of sums and / or differences of literal numbers, any number separated from the rest by plus sign is called a 'term' of the expression. For instance, in the expression 9a + 5b, the terms are 9a and 5b. In the expression 3x–4y–2xy. The terms are 3x, –4y and –2xy.

1.5 Numerical Coefficient :

If a term in an expression is the product of a specific number and a literal number, the specific number is called the numerical coefficient of the term. For example, in the expression 3x–5y, 3 is called the numberical coefficient of the term 3x and (–5) is called the numerical coefficient of the term –5y.

1.6 Like terms and Unlike terms :

In an algebraic expression, terms whose literal numbers are identical are called 'like terms' and terms whose literal numbers are different are called 'unlike terms'.

Consider the expression,

$5x + 7y - 8x + 2y.$

In this expression 5x and –8x are like terms; similarly, 7y and 2y are like terms; on the other hand, 5x an 7y are unlike terms. Similarly, 5x and 2y, –8x and 7y, –8x and 2y are unlike terms.

1.7 Constant term :

In an algebraic expression, the term consisting of no literal number and only a specific number is called a constant. For example, in the expression 2x+3, the term 3 is called the constant term.

Check your Progress

2. What is numerical co-efficient?

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1.8 Evaluating an Expression :

When the literal numbers are given specific numerical values, the process of finding out the value of the algebraic expression is called evaluating the expression. For example, let us consider the expression 3x+2y+5 and evaluate its value when x = 2 and y = 5.

For the given values, we have

$$\begin{aligned} 3x + 2y + 5 &= 3 \times 2 + 2 \times 5 + 5 \\ &= 6 + 10 + 5 \\ &= 21 \end{aligned}$$

Example :

Evaluate the expression $\frac{a+b}{a-b}$, if a = 5 and b = –3.

Putting a = 5 and b = –3 in the given expression we get

$$\frac{a+b}{a-b} = \frac{5-3}{5-(-3)} = \frac{2}{5+3} = \frac{2}{8} = \frac{1}{4}$$

Example :

Evaluate (5x+3y)(x–2y) if x = 4 and y = 5.

$$\begin{aligned} (5x+3y)(x-2y) &= (5\times4+3\times5)(4-2\times5) \\ &= (20+15)(4-10) \\ &= 35\times(-6) = -210 \end{aligned}$$

1.9 Addition of Algebraic Expressions :

Consider the two expressions (5x+3y) and (7x–10y). To add these two expressions, the numerical coefficients of the like terms are added and the respective sums are written as the coefficients in the corresponding like terms in the result. That is,

$$\begin{aligned} (5x+3y) + (7x-10y) &= 5x+3y+7x-10y \\ &= 5x+7x+3y-10y \\ &= (5+7)x+(3-10)y \\ &= 12x-7y. \end{aligned}$$

Some more illustrations are given below.

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Example :

Add the expressions, $5x-3y+6$ and $2x+y-4$.

To perform the operation of addition, it is better we write the expressions one below the other so that coefficients can be added easily.

$$\begin{array}{l} 5x - 3y + 6 \\ 2x + \ \ y - 4 \\ \hline (5+2)x + (-3+1)y + (6-4) \\ \hline 7x - 2y + 2 \end{array}$$

Example :

Add the expressions $2a + 8b - 9$ and $3a - 4b + 11$

$$\begin{array}{l} 2a + 8b - \ \ 9 \\ 3a - 4b + 11 \\ \hline 5a + 4b + \ \ 2 \\ \hline \end{array}$$

Example :

Add $(4r - 9s - 18t)$ and $(8r - 2s - 13t)$

$$\begin{array}{r} 4r - \ \ 9s - 18t \\ 8r - \ \ 2s - 13t \\ \hline 12r - 11s - 31t \\ \hline \end{array}$$

1.10 Subtraction of algebraic expressions :

Suppose the expression $(7x-10y)$ is to be subtracted from the expression $(5x+3y)$. This subtraction is effected by first changing the signs of the numberical coefficients in $(7x-10y)$ and then adding the coefficients of the like terms in the two expressions. That is,

$$= (5x+3y)-(7x-10y)$$

$$= (5x+3y)+(-7x+10y)$$

$$= 5x+3y-7x+10y$$

$$= 5x-7x+3y+10y$$

$$= (5-7)x+(3+10)y$$

$$= -2x+13y.$$

Subtraction operation with expressions is illustrated further below.

Example :

Subtract 2x+y–4 from 5x–3y+6. To subtract, the first step is to change the sign before each term in the expression 2x+y–4 and then the two expressions are to be added.

$$\begin{array}{r} 5x - 3y + 6 \\ -2x - y + 4 \\ \hline 3x - 4y + 10 \\ \hline \end{array}$$

Example :

Subtract (3a–4b+11) from (2a+8b–9)

$$\begin{array}{r} 2a + 8b - 9 \\ -3a + 4b - 11 \\ \hline -a + 12b - 20 \\ \hline \end{array}$$

Example :

Subtract (8r–2s–13t) from (4r–9s–18t)

$$\begin{array}{r} 4r - 9s - 18t \\ -8r + 2s + 13t \\ \hline -4r - 7s - 5t \\ \hline \end{array}$$

1.11 Multiplication of Expressions :

First let us consider the multiplication of a single term expression with another single term expression. Multiplication is effected by adding exponents of the same base in the two expressions. For example, $ab \times a^2b^3$

$$= (a^{1+2})(b^{1+3})$$

$$= a^3b^4$$

Suppose we want to multiply a single term expression by an expression with two or more terms. For example, we want to find out the product of a and (b+c). To find out their product, the single term expression is multiplied by each term in the second expression and the products are added.

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That is, $a \times (b + c) = (a \times b) + (a \times c) = ab + ac$.

Suppose both the expressions have two or more terms. In such situations, taking one term at a time from the first expression, the second expression is multiplied by the terms in the first expression and the products are added. For example,

$$\begin{aligned}(a+b)\,(c+d) &= a(c+d)+b(c+d)\\ &= ac+ad+bc+bd\end{aligned}$$

Note : Taking one term at a time from the second expression the first expression can also be multiplied and the same result may be obtained. That is,

$$\begin{aligned}(a+b)(c+d) &= (a+b)c+(a+b)d\\ &= ac+bc+ad+bd.\end{aligned}$$

In all the examples given above, the terms had no numerical coefficients. But usually, it will not be the case. When terms contain coefficients, in finding out the product of the terms, the coefficients are multiplied and their product is written as the coefficient in the result. For example,

(i) $5a\times 3b = (5\times 3)ab = 15ab$

(ii) $2a(3b+4c) = (2a\times 3b)+(2a\times 4c)$

$= (2\times 3)ab + (2\times 4)ac$

$= 6ab + 8ac$

Example :

Simplify the following expressions :

(i)	$x^2 \times x^7$	(ii)	$(a^2b^3)\,(b^4)$
(iii)	$3ab\times 6a^2b^4$	(iv)	$2\times(5x+3)$
(v)	$(4a+2)(3a+6)$	(vi)	$(x+y)(x+2)$
(vii)	$(x+4)(x+6)$	(viii)	$(5-3a)(1-2a)$
(ix)	$(x-3)(x^2+3x+9)$	(x)	$(2y+5)(3y^2-y+4)$

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Answers

(i) $x^2 \times x^7 = x^{2+7} = x^9$

(ii) $(a^2b^3)(b^4) = a^2 \times b^{3+4} = a^2b^7$

(iii) $3ab \times 6a^2b^4 = (3\times6)(a^{1+2})(b^{1+4})$

$= 18a^3b^5$

(iv) $2x(5x+3) = 2x\times5x+2x\times3$

$= (2\times5)x^{1+1}+(2\times3)(x)$

$= 10x^2+6x$

(v) $(4a+2)(3a+6) = 4a(3a+6)+2(3a+6)$

$= 4a\times3a+4a\times6+2\times3a+2\times6$

$= 12a^2+24a+6a+12$

$= 12a^2+30a+12$

(vi) $(x+y)(x+2) = x(x+2)+y(x+2)$

$= x\times x + x\times2 + y\times x + y\times2$

$= x^2+2x+xy+2y$

(vii) $(x+4)(x+6) = x(x+6)+4(x+6)$

$= x\times x + x\times6 + 4\times x + 4\times6$

$= x^2+6x+4x+24$

$= x^2+10x+24$

(viii) $(5-3a)(1-2a) = 5(1-2a)-3a(1-2a)$

$= (5\times1)+(5)(-2a)+(-3a)(1)+(-3a)(-2a)$

$= 5-10a-3a+6a^2$

$= 5-13a+6a^2$

(ix) $(x-3)(x^2+3x+9) = x(x^2+3x+9)-3(x^2+3x+9)$

$= x^3+3x^2+9x-3x^2-9x-27$

$= x^3-27$

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(x) $\quad (2y+5)(3y^2-y+4) = 2y(3y^2-y+4)+5(3y^2-y+4)$

$$= 6y^3-2y^2+8y+15y^2-5y+20$$

$$= 6y^3+13y^2+3y+20.$$

1.12 Division of Expressions :

For the division of expressions, the law of exponents in division is used.

First let us consider the division of a single term expression by another single term expression. For example, to divide a^4b^2 by ab^5 we perform the operation as follows :

$$\frac{a^4b^2}{ab^5} = \frac{a^{4-1}}{b^{5-2}} = \frac{a^3}{b^3}$$

Let us consider some more examples.

$$\frac{y^6}{y^2} = y^{6-2} = y^4$$

$$\frac{b^3}{b^8} = \frac{1}{b^{8-3}} = \frac{1}{b^5}$$

$$\frac{10a^3b^4}{15ab^2} = \frac{10}{15}\times a^{3-1}\, b^{4-2} = \frac{2}{3}a^2b^2$$

Suppose an expression having two or more terms is to be divided by a single term expression. For example, suppose $5x^5+3x^3+2$ is to be divided by x^2. Here the division operation is performed by taking each term of the expresion in the numerator. That is,

$$\frac{5x^5+3x^3+2}{x^2} = \frac{5x^5}{x^2}+\frac{3x^3}{x^2}+\frac{2}{x^2}$$

$$= 5x^{5-2}+3x^{3-2}+\frac{2}{x^2}$$

$$= 5x^3+3x+\frac{2}{x^2}$$

Note : When the division is not "exact", some terms of the result will remain in fractional form.

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Example :

Simplify the following.

(i) $\dfrac{6a^{13}b^9}{45a^{10}b^4}$

(ii) $\dfrac{22x^5y^2}{55x^3y^{10}}$

(iii) $\dfrac{3y^2+7y}{y}$

(iv) $\dfrac{6a^5-9}{3a^2}$

(v) $\dfrac{6x^3+3x^2-8x+12}{12x}$

Answers :

(i) $\dfrac{6a^{13}b^9}{45a^{10}b^4} = \dfrac{6}{45}\times a^{13-10}b^{9-4} = \dfrac{2}{15}a^3b^5$

(ii) $\dfrac{22x^5y^2}{55x^3y^{10}} = \dfrac{22}{55}\times\dfrac{x^{5-3}}{y^{10-2}} = \dfrac{2x^2}{5y^8}$

(iii) $\dfrac{3y^2+7y}{y} = \dfrac{3y^2}{y}+\dfrac{7y}{y} = 3y+7$

(iv) $\dfrac{6a^5-9}{3a^2} = \dfrac{6a^5}{3a^2}-\dfrac{9}{3a^2} = 2a^3-\dfrac{3}{a^2}$

(v) $\dfrac{6x^3+3x^2-8x+12}{12x} = \dfrac{6x^3}{12x}+\dfrac{3x^2}{12x}-\dfrac{8x}{12x}+\dfrac{12}{12x}$

$= \dfrac{x^2}{2}+\dfrac{x}{4}-\dfrac{2}{3}+\dfrac{1}{x}$

2. EQUATION

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2.1 Meaning

Consider the algebraic expression x+2. Verbally stated, this expresion means 'the sum of a literal number, x and a specific number, 2'. This statement is an incomplete statement. Instead of this, consider the statement x+2=5 which stands for "the sum of a literal number, x and a specific number, 2 is equal to 5. Now, this statement is a complete statement and it is a statement of equality between two quantities. Such a statement is called 'an equation'.

2.2 Identities and Conditional Equations :

In some equations, the statements may be true for any value given to the literal numbers. For example, consider the equation,

$$a(b+c) = ab + ac$$

Put a = 2, b = 3 and c = 5

Now the value of the expression on the left hand side of the above equation is

$$a(b+c) = 2(3+5) = 2(8) = 16$$

The value of the expression on the right hand side of the equation i

$$ab + ac = 2\times3 + 2\times5$$

$$= 6+10 = 16$$

Thus, the left hand side value is equal to the right hand side value. This is true for any set of values given to a, b and c.

This kind of equations which is true for any value given to the literal number is called an identity.

On the other hand, consider the equation x+2 = 5. Here the statemetn is true only for one particular value given for x. That is, only when x takes the value 3, x+2=3+2=5. For no other value this statement is true. Thus, a condition is imposed on the literal number of the equation for the statement to be true. Hence, such an equation is called a 'Conditional equation'.

Conditional equations are the widely used equations and the term 'equation' is generally used to refer to a conditional equation rather than an identity. Therefore, in our lessons, we use the term equation to refer to a conditional equation only.

Check your Progress

3. What is meant by equation?

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2.3 Types of equations :

The literal numbers used in an equation are also known as variables. An equation may contain only one variable or two variables or more than two variables. We are concerned with equations in one variable and equations in two variables.

I. Examples of equations in one variable

(i) $2x+5 = 7$

(ii) $3x-4 = 2x+1$

(iii) $\frac{x-5}{2} = \frac{4x+1}{5}$

(iv) $2x^2-5x+6 = 0$

(v) $7x^4+3x^3+5x^2-2x+1 = 0$

II. Examples of equations in two variables :

(i) $2x+3y = 7$

(ii) $5x^2+6xy+7y^2 = 0$

(iii) $x^3+y^3 = 6$

(iv) $2x^5+3x^3y+5x^3y^2+8x^2y^3+6xy^4+y^5 = 0$

In the first three examples of equations in one variable and in the first example of equations in two variables, the highest index of variable is 1. In all the other examples, the highest index of variable is 2 or more than 2.

When the highest index of variable is one, the equation is called a simple equation. If the highest index is equal to 2, the equation is called a quadratic equation. If the highest index is equal to 3, it is called a cubic equation. In general, if the highest index is n, then the equation is called n^{th} degree equation.

We are going to study about only simple equations in one variable and in two variables, and quardratic equations in one variable.

3. SOLVING EQUATIONS

3.1 Solving simple equations in one variable

As already explained any equation contains some variables whose values are not known. When we put some value to the unknown variable, if the left hand side (L.H.S.) becomes equal to the right hand side (R.H.S.) of

the equation, we say that the particular value is the 'root' or 'solution' of the equation. The method of finding out the root is called 'solving the equation'.

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In solving the equation, we employ certain operations that preserve equality and lead us to a proposed value of the variable. The operations performed are :

(i) **Addition and Subtraction :** The same quantity may be added to or subtracted from both sides of an equation.

(ii) **Multiplication and Division :** Both sides of an equation may be multiplied or divided by the same quantity (except zero).

The method of solving an equation using these operations is illustrated below in the case of simple equations in one variable.

Example

Solve the equation $x+5 = 15$

Subtracting 5 from both sides of the equation, we get

$$x+5-5 = 15-5$$

$$\therefore x = 10$$

We have found out the value of the unknown variable. That is, we have solved the given equation. The root of the equation is 10.

Example

Solve the equation $x-7 = 2$

Adding 7 to both sides, we get

$$x-7+7 = 2+7$$

$$\therefore x = 9$$

9 is the root of the equation.

Example

Solve the equation $7x = 14$

Dividing bothsides by 7, we get

$$\frac{7x}{7} = \frac{14}{7}$$

$$\therefore x = 2$$

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Example

Solve $$\frac{x}{3} = 11$$

Multiplying both sides by 3, we get

$$\frac{x}{3} \times 3 = 11 \times 3$$

$$\therefore \quad x = 33$$

Example

Solve $$\frac{1}{2}x + 4 = 5$$

Subtracting 4 from both sides, we get

$$\frac{1}{2}x + 4 - 4 = 5 - 4$$

$$\therefore \quad \frac{1}{2}x = 1$$

Now, multiplying both sides by 2, we get,

$$2 \times \frac{1}{2}x = 1 \times 2$$

$$\therefore \quad x = 2$$

2 is the root of the equation.

Example

Solve $$4x - 5 = 8x - 17$$

Adding 5 on bothsides, we get

$$4x - 5 + 5 = 8x - 17 + 5$$

$$4x = 8x - 12$$

Subtracting 8x on bothsdies, we get

$$4x - 8x = 8x - 12 - 8x$$

$$-4x = -12$$

Dividing both sides by (–4), we get

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$$\frac{-4x}{4} = \frac{-12}{-4}$$

$$\therefore \quad x = 3$$

3 is the root of the equation.

Note : In the problems of solving the equation, the answer may be checked by putting the answer in the place of x in the given equation and by verifying whether L.H.S. = R.H.S.

For the above example, we check the answer as follows.

Putting $x = 3$, we get

$$\begin{aligned} \text{L.H.S.} &= 4x-5 \\ &= 4\times3-5 \\ &= 12-5 = 7 \\ \text{R.H.S.} &= 8x-17 \\ &= 8\times3-17 \\ &= 24-17 = 7 \\ \therefore \text{L.H.S.} &= \text{R.H.S.} \end{aligned}$$

$\therefore$ Answer is correct.

Example :

Solve $\quad \frac{8x+3}{6}+\frac{1}{2} = \frac{7x-1}{4}$

Here, L.C.M. (Least Common Multiple) of the three numbers in the denominators namely 6, 2, 4 is found out and both sides of the equation are multiplied by it.

L.C.M. of 6, 2, 4 is 12.

$\therefore$ Multiply both sides by 12.

$$\begin{aligned} \frac{8x+3}{6}\times12+\frac{1}{2}\times12 &= \frac{7x-1}{4}\times12 \\ (8x+3)2+6 &= (7x-1)3 \\ 16x+6+6 &= 21x-3 \\ 16x+12 &= 21x-3 \end{aligned}$$

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$$16x+12-12 = 21x-3-12$$
$$16x = 21x-15$$
$$16x-21x = 21x-15-21x$$
$$-5x = -15$$
$$\frac{-5x}{-5} = \frac{-15}{-5}$$
$$\therefore \; x = 3$$

Check : Substitute x = 3 in the equation

$$\text{L.H.S.} = \frac{8\times 3+3}{6}+\frac{1}{2}$$
$$= \frac{24+3}{6}+\frac{1}{2}$$
$$= \frac{27}{6}+\frac{1}{2} = \frac{27+3}{6}$$
$$= \frac{30}{6} = 5$$
$$\text{R.H.S.} = \frac{7\times 3-1}{4} = \frac{21-1}{4} = \frac{20}{4} = 5$$
$$\text{L.H.S.} = \text{R.H.S.}$$

$\therefore$ x = 3 satisfies the equation.

3.2 Solving Quadratic Equations in one variable

3.2.1 Meaning, Definition and Types of Quadratic Equations

As we have already stated, if the highest power of the variable is 2, the equation is called a quadratic equation. The general form of a quadratic equation is given as $ax^2+bx+c = 0$.

In this equation, x is the variable and a, b, c are the numerical coefficients. Depending upon the values of a, b, c quadratic equations may be classified into two types. They are :

(1) Pure quadratic equation (2) Adfected quadratic equation

If the numerical coefficient of x term is zero, the equation is called pure quadratic equation. Its general form is given by

$$ax^2+c = 0.$$

Some examples of pure quadratic equation are :

(i) $7x^2-175 = 0$

(ii) $x^2+20 = 0$

(iii) $x^2-9a^2 = 0$

(iv) $2x^2-3a^2 = x^2+a^2$

If the numerical coefficient of x term is not equal to zero the equation is called 'adfected quadratic equation'. In an adfected quadratic equation, the constant term c may be equal to zero or may not be equal to zero. The general forms of this type of equation are.

$$ax^2+bx = 0$$

$$ax^2+bx+c = 0$$

Some examples are given below :

(i) $x^2-5x = 0$

(ii) $2x^2+7x = 0$

(iii) $rx^2-sx = 0$

(iv) $x^2+5x+6 = 0$

(v) $4x^2+20x+25 = 0$

Now let us see the methods of solving these different types of quadratic equations.

3.2.2 Solving pure quadratic equations :

$ax^2+c = 0$ is the general form of this type of equation. To solve this equation, we follow the following steps.

Subtracting c from bothsides, we get

$$ax^2 = -c$$

Dividing bothsides by a, we get

$$x^2 = \frac{-c}{a}$$

$$\therefore x = +\sqrt{\frac{-c}{a}} \text{ or } -\sqrt{\frac{-c}{a}} = \pm\sqrt{\frac{-c}{a}}$$

$+\sqrt{\frac{-c}{a}}$ and $-\sqrt{\frac{-c}{a}}$ are the roots of the given equation.

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Check your Progress

4. Define quadratic equation.

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Example

Solve the equation $7x^2-175 = 0$

$$7x^2-175 = 0$$

$$7x^2 = 175$$

$$\therefore \quad x^2 = \frac{175}{7} = 25$$

$$\therefore \quad x = \pm\sqrt{25} = \pm 5.$$

$\therefore$ 5, –5 are the roots of the given equation.

Example

$3x^2+20 = 2x^2+11$. Find the roots of this equation

$$3x^2+20 = 2x^2+11$$

$$3x^2-2x^2 = 11-20$$

$$x^2 = -9$$

$$\therefore \quad x = \pm\sqrt{-9} = \pm 3\sqrt{-1}$$

$\sqrt{-1}$ is called imaginary number and is denoted by i.

$$\therefore \quad x = \pm 3i$$

$\therefore$ 3i and –3i are the roots of the given equation.

Example

Solve $\quad x^2+20 = 56$

$$x^2+20 = 56$$

$$x^2 = 56-20$$

$$\therefore \quad x^2 = 36$$

$$\therefore \quad x = \pm\sqrt{36} = \pm 6$$

6, –6 are the roots of the equation.

Example

Solve $\quad 2x^2-3a^2 = x^2+a^2$

$$2x^2-3a^2 = x^2+a^2$$

$$2x^2-x^2 = a^2+3a^2$$

$$x^2 = 4a^2$$

$$\therefore \quad x = \pm\sqrt{4a^2} = \pm 2a$$

2a, –2a are the roots of the equation.

2.2.3 Solving adfected quadratic equations :

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Case (i) :

Suppoe c=0. Then the general form of the quadratic equation becomes

$$ax^2+bx = 0$$

This equation may be written as,

$$x(ax+b) = 0$$

The product is zero only if either x = 0 or (ax+b) = 0

That is, $x = 0$ or $x = \frac{-b}{a}$

$\therefore$ 0 and $\frac{-b}{a}$ are the roots of this type of equation.

Example

$$x^2-4x = 0. \quad \text{Find the roots.}$$

$$x^2-4x = 0$$

(i.e.) $$x(x-4) = 0$$

$$\therefore \quad x = 0 \quad \text{or } x-4 = 0$$

$$x = 0 \quad \text{or} \quad x = 4$$

0 and 4 are the roots of the given equation.

Example

Solve $$2x^2+7x = 0$$

$$2x^2+7x = 0$$

$$x(2x+7) = 0$$

$$x = 0 \text{ or } 2x+7 = 0$$

$$x = 0 \text{ or } x = -7/2$$

0 and –7/2 are the roots of the given equation.

Case (ii) :

When the numerical coefficients and the constant term are not equal to zero, the quadratic equation takes the form $ax^2+bx+c = 0$ where $a\neq0$, $b\neq0$, $c\neq0$ ($\neq$ is read as 'is not equal to').

Here the equation can be solved using any one of the following three methods :

Space for hints

(a) Factorization method

(b) Method of square

(c) General method

(a) Factorization Method :

To understand this method first let us see some examples.

Consider the equation,

$$x^2 + 5x + 6 = 0$$

This equation may be rewritten as $x^2+3x+2x+2\times3 = 0$. Taking the common factor x outside from the first two terms and 2 from the last two terms, we get

$$x(x+3)+2(x+3) = 0$$

Again taking the common factor (x+3) outside, we get

$$(x+3)(x+2) = 0$$

This equation is true only when

$$(x+3) = 0 \text{ or } (x+2) = 0$$

$$\text{That is, } x = -3 \text{ or } x = -2$$

∴ –3 and –2 are the roots of the equation.

Example

Solve $\quad 2x^2+5x-3 = 0$

$$2x^2+5x-3 = 0$$

$$2x^2+6x-x-3 = 0$$

$$2x(x+3)-(x+3) = 0$$

$$(x+3)(2x-1) = 0$$

$$x+3 = 0 \text{ or } (2x-1) = 0$$

$$x = -3 \text{ or } x = +1/2$$

∴ –3 and $\frac{1}{2}$ are the roots.

In all these examples, we tried to express the numerical coefficient of the term containing x as a sum or difference between the factors of the product of the numerical coefficient of x^2 term and constant term. In the first example, product of the coefficient of x^2 and constant term = $1\times6 = 6$. For this product, 2 and 3 are two factors of 6. That is $2\times3 = 6$. Now, the

coefficient of x (which is 5) is expressed as a sum of these two factors and simplied to get the L.H.S. as a product of two expressions so that each expression can be equated to zero and get the value of x.

Space for hints

In the second example, the product of the coefficient of x^2 and constant term is $2x(-3) = -6$. Two factors of -6 are 6 and (-1). That is, $-6 = 6\times(-1)$. Now, the coefficient of x (=5) may be expressed as the sum of these two factors (that is, $5 = 6+(-1) = 6-1$ and simplified to get the values of x.

Steps in the factorization method

(i) The coefficient of x is to be written as a sum or difference between any two factors of the product obtained by multiplying the coefficient of x^2 and the constant term in the given equation.

(ii) After splitting the x term into two terms as mentioned above, the L.H.S. expression is factorised. That is, it is rewritten as a product of two expressions called factors. Now, each factor is equated to zero and the values of x are obtained.

In order to better understand the factorization method, we have given below some more examples.

Example 1 :

Solve the equation.

$$7x^2-11x-6 = 0$$

In the given equation, coefficient of $x^2 = 7$ constant $= -6$

$$7\times(-6) = -42$$

(-14) and 3 are two factors of -42. That is, $(-14)\times3=-42$

If we add these two factors, we get the coefficient of x as the sum. That is,

$$(-14)+(3) = -14+3 = -11$$

$\therefore$ The given equation can be rewritten as follows :

$$7x^2-14x+3x-6 = 0$$

$$(7x)(x)-(7x)(2)+(3)(x)-(3)(2) = 0$$

$$\therefore 7x(x-2)+3(x-2) = 0$$

$$(x-2)(7x+3) = 0$$

$$x-2 = 0 \text{ or } 7x+3 = 0$$

$$x = 2 \text{ or } x = -3/7$$

Space for hints

Example 2 :

$$5x^2+2x-3 = 0. \text{ Solve}$$

$$(5)\times(-3) = -15$$

$$5-3 = 2$$

∴ The given equation can be written as,

$$5x^2+5x-3x-3 = 0$$

$$5x(x+1)-3(x+1) = 0$$

$$(x+1)(5x-3) = 0$$

$$\therefore x+1 = 0 \text{ or } 5x-3 = 0$$

$$x = -1 \text{ or } x = 3/5$$

Example 3 :

$$x^2-x-12 = 0. \text{ Solve}$$

$$(1)x(-12) = -12$$

$$(-12) = (-4)\times(3)$$

$$-4+3 = -1$$

∴ The given equation can be rewritten as

$$x^2-4x+3x-12 = 0$$

$$x(x-4)+3(x-4) = 0$$

$$(x-4)(x+3) = 0$$

$$(x-4) = 0 \quad \text{or } (x+3) = 0$$

$$\therefore x = 4 \text{ or } -3$$

Example 4 :

$$y^2-9y+18 = 0$$

$$1\times18 = 18$$

$$18 = (-6)\times(-3)$$

$$(-6)+(-3) = -6-3 = -9$$

∴ The given equation is written as

$$y^2-6y-3y+18 = 0$$

$$y(y-6)-3(y-6) = 0$$

Space for hints

$$(y-6)(y-3) = 0$$

$$y-6 = 0 \quad \text{or } y-3 = 0$$

$$y = 6 \text{ or } 3$$

(b) Method of Square

In this method, the LHS expression is rewritten as a perfect square of the type $(a+b)^2$ or $(a-b)^2$. The expansions for $(a+b)^2$ and $(a-b)^2$ are as follows:

$$(a+b)^2 = a^2+2ab+b^2$$

$$(a-b)^2 = a^2-2ab+b^2$$

The term of the LHS expression is rewritten as a perfect square of the type $(a+b)^2$ or $(a-b)^2$. the expansions for $(a+b)^2$ and $(a-b)^2$ are as follows :

$$(a+b)^2 = a^2+2ab+b^2$$

$$(a-b)^2 = a^2-2ab+b^2$$

The term of the LHS expression of the given equation are compared with the terms of the expansions of $(a+b)^2$ and $(a-b)^2$. If they correspond to terms in anyone of these two squares, the respective square form is used to rewrite the given expression. Then, the base of the square is equated to zero. In this case, we will get only one value for x, implying that the two roots of x are equal. The square method is illustrated below with examples.

Example 1 :

$$4x^2+20x+25 = 0. \text{ Solve}$$

The LHS expression can be rewritten as follows.

$$(2x^2)+2(2x)(5)+(5)^2 = 0.$$

Let $a = 2x$ and $b = 5$. Now, the LHS expression becomes, $a^2+2ab+b^2 = 0$

$$\therefore\ (a+b)^2 = 0$$

Replacing a and b, we get

$$(2x+5)^2 = 0$$

$2x+5$ is the base. Equating it to zero,

$$2x+5 = 0.$$

$$\therefore\ x = -5/2$$

Space for hints

Example 2:

Solve $x^2-12x+36 = 0$

Given equation can be rewritten as

$$(x)^2-2(x)(6)+(6)^2 = 0$$

Let a = x and b = 6.

$$\therefore\ a^2-2ab+b^2 = 0$$

$$(a-b)^2 = 0$$

That is, $(x-6)^2 = 0$

$$\therefore\ x-6 = 0 \quad \text{or } x = 6$$

Example 3:

Solve $9x^2+48x+64 = 0$

$$9x^2+48x+64 = 0$$

$$(3x)^2+2(3x)(8)+(8)^2 = 0$$

$$(3x+8)^2 = 0$$

$$3x+8 = 0 \qquad x = -8/3.$$

Example 4:

Solve $16y^2-48y+36 = 0$

The given equation can be rewritten as

$$(4y)^2-2(4y)(6)+(6)^2 = 0$$

$$(4y-6)^2 = 0$$

$$\therefore\ 4y-6 = 0 \quad \text{or} \quad y = 6/4 = 3/2$$

(c) General Method :

In each and every problem considered so far, LHS expression has been expressed as a product of two factors or as a perfect square. But, it may not be possible in some situations. In such situations, a general method is used to solve the given equation. The general method is explained below :

The general form of the quardratic equation is $ax^2+bx+c = 0$

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Here the solution is given by the formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

That is, the roots of the equation are,

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Example 1:

Solve $x^2 - 8x + 7 = 0$

For the equation $ax^2+bx+c = 0$, the roots are

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

From the given equation, we get

$$a = 1$$
$$b = -8$$
$$c = 7$$

Putting these values of a, b, c in the formula for roots we get,

$$\frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(7)}}{2 \times 1} = \frac{8 \pm \sqrt{64 - 28}}{2}$$

$$= \frac{8 \pm \sqrt{36}}{2} = \frac{8 \pm 6}{2}$$

$$= \frac{8+6}{2} \text{ or } \frac{8-6}{2} = \frac{14}{2} \text{ or } \frac{2}{2} = 7 \text{ or } 1$$

∴ 7 and 1 are the roots of the given equation.

Example 2:

Solve $2x^2 - 10x - 1 = 0$

From the given equation, we get

$a = 2$; $b = -10$; $c = -1$

The roots are $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{-(-10) \pm \sqrt{(-10)^2 - 4(2)(-1)}}{2 \times 2}$

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$$= \frac{10 \pm \sqrt{100+8}}{4} = \frac{10 \pm \sqrt{108}}{4} = \frac{10 \pm \sqrt{3 \times 36}}{4}$$

$$= \frac{10 \pm 6\sqrt{3}}{4} = \frac{10+6\sqrt{3}}{4}, \frac{10-6\sqrt{3}}{4}$$

$$= \frac{5+3\sqrt{3}}{2}, \frac{5-3\sqrt{3}}{2}$$

Example 3:

Solve $x^2+x+1 = 0$

Here a = 1, b = 1, c = 1.

$$\therefore \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-1 \pm \sqrt{1^2 - 4 \times 1 \times 1}}{2 \times 1}$$

$$= \frac{-1 \pm \sqrt{1-4}}{2}$$

$$= \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

$\therefore$ The roots are $\frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}$

Example 4:

Solve $9x^2-6x+1 = 0$

a = 9, b = –6, c = 1

$$\therefore \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(9)(1)}}{2 \times 9}$$

$$= \frac{6 \pm \sqrt{36-36}}{18}$$

$$= \frac{6 \pm 0}{18} = \frac{6}{18} = \frac{1}{3}$$

Here, the two roots are equal and the value is $\frac{1}{3}$.

Example 5 :

Sovle the equation

$$\frac{1}{x+7} + \frac{1}{x+3} = \frac{6}{5}$$

Space for hints

In the given equation, the LHS consists of two fractions $\frac{1}{x+7}$ and $\frac{1}{x+3}$. To add these two fractions, we consider the LCM of (x+7) and (x+3). The LCM is (x+7). (x+3) and we take it as the common denominator and add the fractions as follows.

$$\frac{1}{x+7}+\frac{1}{x+3} = \frac{(x+3)+(x+7)}{(x+7)(x+3)}$$

$$= \frac{2x+10}{x^2+3x+7x+21} = \frac{2x+10}{x^2+10x+21}$$

Now, the given equation can be rewritten as

$$\frac{2x+10}{x^2+10x+21} = \frac{6}{5}$$

Cross multiplying, we get

$$5(2x+10) = 6(x^2+10x+21)$$

$$10x+50 = 6x^2+60x+126$$

$$6x^2+60x+126-10x-50 = 0$$

$$6x^2+50x+76 = 0$$

This equation resembles the general equation,

$ax^2+bx+c = 0$. Now, a = 6, b = 50, c = 76.

$$\therefore \quad x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$= \frac{-50\pm\sqrt{50^2-4\times 6\times 76}}{2\times 6} = \frac{-50\pm\sqrt{2500-1824}}{2\times 6}$$

$$= \frac{-50\pm\sqrt{676}}{12} = \frac{-50\pm 26}{12} = \frac{-50+26}{12}, \frac{-50-26}{12}$$

$$= \frac{-24}{12}, \frac{-76}{12} = -2, \frac{-19}{3}$$

$\therefore$ The roots of the given equation are –2 and –19/3.

Space for hints

Example 6:

Solve, $\dfrac{2}{x+5}+\dfrac{3}{2x+1} = \dfrac{7}{8}$

In the given equation,

$$\text{LHS} = \frac{2}{(x+5)}+\frac{3}{(2x+1)}$$

$$= \frac{2(2x+1)+3(x+5)}{(x+5)(2x+1)}$$

$$= \frac{4x+2+3x+15}{2x^2+10x+x+5} = \frac{7x+17}{2x^2+11x+5}$$

∴ The given equation becomes,

$$\frac{7x+17}{2x^2+11x+5} = \frac{7}{8}$$

$$8(7x+17) = 7(2x^2+11x+5)$$

$$56x+136 = 14x^2+77x+35$$

$$14x^2+77x+35-56x-136 = 0$$

$$14x^2+21x-101 = 0$$

$a = 14$; $b = 21$; $c = -101$

$$\therefore \quad x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-21 \pm \sqrt{21^2-4\times 14(-101)}}{2\times 14}$$

$$= \frac{-21 \pm \sqrt{441+5656}}{28}$$

$$= \frac{-21 \pm \sqrt{6097}}{28}$$

∴ The roots are $\dfrac{-21+\sqrt{6097}}{28}$ and $\dfrac{-21-\sqrt{6097}}{28}$

Example 7:

Solve $3x^2+5 = 4x$

The given equation can be rewritten as

$$3x^2-4x+5 = 0$$

Now, a = 3; b = –4; c = 5.

$$\therefore \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(3)(5)}}{2 \times 3}$$

$$= \frac{4 \pm \sqrt{16-60}}{6} = \frac{4 \pm \sqrt{-44}}{6}$$

$$= \frac{4 \pm 2\sqrt{-11}}{6} = \frac{2\left(2 \pm i\sqrt{11}\right)}{6}$$

$$= \frac{2 \pm i\sqrt{11}}{3} = \frac{2 + i\sqrt{11}}{3}, \ \frac{2 - i\sqrt{11}}{3}$$

Example 8:

$$y^2+10y = 18. \text{ Solve}$$

$$y^2+10y = 18$$

$$\therefore \ y^2+10y-18 = 0$$

a = 1; b = 10; c = –18.

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-10 \pm \sqrt{10^2 - 4(1)(-18)}}{2 \times 1}$$

$$= \frac{-10 \pm \sqrt{100+72}}{2}$$

$$= \frac{-10 \pm \sqrt{172}}{2} = \frac{-10 \pm \sqrt{4 \times 43}}{2}$$

$$= \frac{-10 \pm 2\sqrt{43}}{2} = -5 \pm \sqrt{43}$$

$$= -5+\sqrt{43}, \ -5-\sqrt{43}$$

$\therefore$ The roots of the given equation are $-5+\sqrt{43}$ and $-5-\sqrt{43}$.

Example 9:

Solve

$$2x^2 = 3x+7$$

$$2x^2 = 3x+7$$

$$2x^2-3x-7 = 0$$

Space for hints

Space for hints

∴ a = 2, b = –3, c = –7

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{- -3 \pm \sqrt{(-3)^2 - 4(2)(-7)}}{2 \times 2}$$

$$= \frac{3 \pm \sqrt{9+56}}{4} = \frac{3 \pm \sqrt{65}}{4}$$

∴ The roots are $\frac{3+\sqrt{65}}{4}, \frac{3-\sqrt{65}}{4}$

3.3 Solving Simultaneous Linear Equations in two variables :

So far we have seen simple and quadratic equations in one variable. For the simple equations we got ony one solution and for the quadratic we got two solutions. Now, we are going to solve equations in two variables. First let us consider a simple equation in two variables. If we have only one simple equation in two variables, we will get infinite number of solutions. That is, for each value of x we will get a value for y. For example, consider the equation,

$$2x+3y = 5$$

When x = 1, the equation becomes,

$$2 \times 1+3y = 5$$

$$3y = 5-2 = 3$$

$$y = \frac{3}{3} = 1$$

Thus, (x = 1, y = 1) is a solution of the given equation similarly, whenx=2, $y = \frac{1}{3}$, when x = –3, $y = \frac{11}{3}$ and so on.

$\left(2, \frac{1}{3}\right), \left(-3, \frac{11}{3}\right)$..... are all solutions of the given equation. Thus, there is an infinite number of solutions for the given equation.

Suppose we have two equations in two variables. There will be only one solution common to both the equations except when the corresponding coefficients and the constants in the two equations are proportional. For

example, consider the two equations.

$$2x+3y = 5$$

$$6x+9y = 15$$

Here the coefficients and constants are proportional.

That is, $\frac{2}{6} = \frac{3}{9} = \frac{5}{15}$

If we divide both sides of the second equation by 3, we get the result as identical with the first equation. Therefore, though they appear to be two equations in two variables, actually they represent a single equation in two variables and hence, there will be infinite number of solutions. On the otherhand, if the coefficients and constants of the given equations are not proportional, there will be only one solution common to both the equations. Now, our problem is to find that unique common solution to the given pair of equations in two variables. Finding such a solution is called 'solving simultaneous linear equations in two variables'.

Space for hints

3.3.1 The simplest method of solving Simultaneous Equations

Procedure :

1. get from one equation, an expression of one variable say, y in terms of the other variable x
2. substitute this expression for y into the second equation
3. solve the second equation which is in terms of only one variable namely, x and
4. substitute the solution for x in the expression for y and thus get the solution for y

The following examples will illustrate the method

$$x+y-3 = 0 \qquad \text{------(1)}$$

$$x-3y+1 = 0 \qquad \text{------(2)}$$

From (1), $y = 3-x$.

Putting $y = 3-x$ in (2), we get,

$$x-3(3-x)+1 = 0$$

$$x-9+3x+1 = 0$$

$$4x-8 = 0$$

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$$\therefore \quad x = \frac{8}{4} = 2$$

$$\therefore \quad y = 3-x = 3-2 = 1$$

$x = 2$ and $y = 1$ is the unique solution to the given equations.

Note : You can check your answer by putting $x = 2$ and $y = 1$ in anyone of the given equations and by verifying whether L.H.S. = R.H.S.

Example 1:

Solve the following equations,

$$x+5y = -1$$

$$2x+7y = 1$$

From the first equation, $x = -1-5y$. Putting this expression for x in the second equation, we get,

$$2(-1-5y)+7y = 1$$

$$-2-10y+7y = 1$$

$$-3y = 1+2 = 3$$

$$\therefore \quad y = \frac{3}{-3} = -1$$

$$\therefore \quad x = -1-5y$$

$$= -1-5(-1)$$

$$= -1+5 = 4$$

$x = 4$ and $y = -1$ is the solution of the given equations.

Example 2:

Solve $x+2y-3=0$ and $16x-20y-13 = 9$.

$$x+2y-3 = 0 \qquad \text{------(1)}$$

$$16x-20y-13 = 9 \qquad \text{------(2)}$$

From (1), $x = 3-2y$. Putting this in (2),

$$16(3-2y)-20y-13 = 9$$

$$48-32y-20y-13-9 = 0$$

$$26-52y = 0$$

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$$\therefore \quad y = \frac{-26}{-52} = \frac{1}{2}$$

$$\therefore \quad x = 3-2y = 3-2\times\frac{1}{2} = 3-1 = 2$$

$\therefore \left(x = 2,\ y = \frac{1}{2}\right)$ is the solution for the given equations.

3.3.2. Method of Elimination :

Some times an alternative method called 'the method of elimination' is used to solve the given simultaneous equations. In this method, anyone of the given equations or both the equations are multiplied by appropriate numbers such that the numerical coefficient of anyone of the two variables become equal in both the equations. After doing this, either we add or subtract one equation from another so that one variable is eliminated and we get the result in only the second variable. Value of this variable is obtained by solving this equation and by substitution of this value in anyone of the given equations, the value of the first variable is obtained. The procedure is illustrated below.

$$2x+3y = 12 \qquad \text{------(1)}$$

$$3x+6y = 13 \qquad \text{------(2)}$$

Multiplying equation (1) by 2, we get,

$$4x+6y = 24 \qquad \text{-------(3)}$$

Now, (2) and (3) have the same numerical coefficient for y. If we substract (2) from (3), we get,

$$\begin{aligned} 4x+6y &= 24 \\ -3x-6y &= -13 \\ \hline x &= 11 \end{aligned}$$

Put x = 11 in (1).

$$\begin{aligned} 2\times11+3y &= 12 \\ 3y &= 12-22 = -10 \\ \therefore \quad y &= -\frac{10}{3} \end{aligned}$$

$\left(x = 11,\ y = -\frac{10}{3}\right)$ is the solution to the given equations.

Space for hints

Example 3:

Solve

$$3x+2y = 5 \quad \text{------(1)}$$

$$x-7y = 15 \quad \text{------(2)}$$

Multiply (2) by 3, we get

$$3x-21y = 45 \quad \text{------(3)}$$

(1) – (3) gives,

$$\begin{aligned} 3x+2y &= 5 \\ -3x+21y &= -45 \\ \hline 23y &= -40 \end{aligned}$$

$$\therefore \quad y = \frac{-40}{23}$$

Put $y = \frac{-40}{23}$ in (2)

$$x - 7\left(\frac{-40}{23}\right) = 15$$

$$x + \frac{280}{23} = 15$$

$$\therefore \quad x = 15 - \frac{280}{23} = \frac{345-280}{23} = \frac{65}{23}$$

$\left(x = \frac{65}{23},\ y = \frac{-40}{23}\right)$ is the solution to the given equations.

Example 4:

Solve the equations

$$3x-4y = 23 \quad \text{------(1)}$$

$$5x+9y = 7 \quad \text{------(2)}$$

Multiply (1) by 5 and (2) by 3. We get,

$$15x-20y = 115 \quad \text{------(3)}$$

$$15x+27y = 21 \quad \text{------(4)}$$

(3)–(4) gives

$$-47y = 94$$

$$\therefore \quad y = \frac{94}{-47} = -2$$

Put $y = -2$ in (1)

$$3x-4(-2) = 23$$

$$3x+8 = 23$$

$$3x = 23-8 = 15$$

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$$\therefore \quad x = \frac{15}{3} = 5$$

(x = 5, y = –2) is the solution of the given equations.

3.3.3 General method or formula method

The general pair of simultaneous linear equations in two variables is given as,

$$a_1x+b_1y+c_1 = 0$$

$$a_2x+b_2y+c_2 = 0$$

For this pair of equations, the solution is given by the following formulae :

$$x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}; \quad y = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$$

We illustrate the method by the following examples.

Example 1:

Solve the equations

$$3x+2y = 13$$

$$5x+3y = 21$$

First we rewrite the given equations to resemble the general form as follows :

$$3x+2y-13 = 0$$

$$5x+3y-21 = 0$$

From these equations we get,

$a_1 = 3 \quad b_1 = 2 \quad c_1 = -13$

$a_2 = 5 \quad b_2 = 3 \quad c_2 = -21$

$$\therefore \quad x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}$$

$$= \frac{2(-21) - 3(-13)}{3 \times 3 - 5 \times 2} = \frac{-42+39}{9-10}$$

$$= \frac{-3}{-1} = 3$$

Space for hints

$$y = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$$

$$= \frac{(-13)5-(-21)3}{3\times3-5\times2} = \frac{-65+63}{9-10}$$

$$= \frac{-2}{-1} = 2$$

∴ $(x = 3, y = 2)$ is the required solution.

Example 2:

Solve the equations $3x-4y = 23$ and $5x+9y = 7$.

The given equations can be rewritten as,

$$3x-4y-23 = 0$$

$$5x+9y-7 = 0$$

$a_1 = 3 \quad b_1 = -4 \quad c_1 = -23$

$a_2 = 5 \quad b_2 = 9 \quad c_2 = -7$

$$x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}$$

$$= \frac{(-4)(-7)-(9)(-23)}{3\times9-(5)(-4)}$$

$$= \frac{28+207}{27+20} = \frac{235}{47} = 5$$

$$y = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$$

$$= \frac{(-23)(5)-(-7)(3)}{3\times9-(5)(-4)}$$

$$= \frac{-115+21}{27+20} = \frac{-94}{47} = -2$$

$(5, -2)$ is the solution to the given equations. (By putting $x = 5$ and $y = -2$ in anyone of the given equations, you can verify your answer).

Example 3:

Solve $\quad 3x+5y = -7$

$\quad 4x-3y = 10$

Space for hints

We can rewrite the given equations as follows :

$$3x+5y+7 = 0$$

$$4x-3y-10 = 0$$

$a_1 = 3 \quad b_1 = 5 \quad c_1 = 7$

$a_2 = 4 \quad b_2 = -3 \quad c_2 = -10$

$$x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}$$

$$= \frac{(5)(-10)-(-3)(7)}{(3)(-3)-(4)(5)}$$

$$= \frac{-50+21}{-9-20} = \frac{-29}{-29} = 1$$

$$y = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$$

$$= \frac{(7)(4)-(-10)(3)}{-29}$$

$$= \frac{28+30}{-29} = \frac{58}{-29} = -2$$

(1, –2) is the required solution.

Example 4:

Solve
$$x+5y+1 = 0$$

$$2x+7y-1 = 0$$

$a_1 = 1 \quad b_1 = 5 \quad c_1 = 1$

$a_2 = 2 \quad b_2 = 7 \quad c_2 = -1$

$$x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}$$

$$= \frac{(5)(-1)-(7)(1)}{(1)(7)-(2)(5)}$$

$$= \frac{-5-7}{7-10} = \frac{-12}{-3} = 4$$

$$y = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$$

Space for hints

$$= \frac{(1)(2)-(-1)(1)}{-3}$$

$$= \frac{2+1}{-3} = \frac{3}{-3} = -1$$

(4, –1) is the required solution.

Example 5:

Solve
$$\frac{3x}{8}+\frac{2y}{15} = \frac{5}{6}$$
$$\frac{x}{2}-\frac{7y}{20} = \frac{15}{4}$$

We rewrite the given equations as follows :

$$\frac{3x}{8}+\frac{2y}{15}-\frac{5}{6} = 0$$
$$\frac{x}{2}-\frac{7y}{20}-\frac{15}{4} = 0$$

$$a_1 = \frac{3}{8} \quad b_1 = \frac{2}{15} \quad c_1 = \frac{-5}{6}.$$
$$a_2 = \frac{1}{2} \quad b_2 = \frac{-7}{20} \quad c_2 = \frac{-15}{4}$$

$$x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}$$

$$= \frac{\left(\frac{\cancel{2}}{\cancel{15}}\right)\left(\frac{-\cancel{15}}{\cancel{4}_2}\right)-\left(\frac{-7}{\cancel{20}_4}\right)\left(\frac{-\cancel{5}}{6}\right)}{\left(\frac{3}{8}\right)\left(\frac{-7}{20}\right)-\left(\frac{1}{\cancel{2}}\right)\left(\frac{\cancel{2}}{15}\right)}$$

$$= \frac{-\frac{1}{2}-\frac{7}{24}}{-\frac{21}{160}-\frac{1}{15}} = \frac{\frac{-12-7}{24}}{\frac{-63-32}{480}}$$

$$= \frac{\frac{-19}{24}}{\frac{-95}{480}} = \frac{19}{24}\times\frac{480}{95} = 4$$

$$y = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$$

$$= \frac{\left(\frac{-5}{6}\right)\left(\frac{1}{2}\right) - \left(\frac{-15}{4}\right)\left(\frac{3}{8}\right)}{\frac{-95}{480}}$$

$$= \frac{\frac{-5}{12} + \frac{45}{32}}{\frac{-95}{480}} = \frac{\frac{-40+135}{96}}{\frac{-95}{480}}$$

$$= \frac{\frac{95}{96}}{\frac{-95}{480}} = \frac{95}{96} \times \frac{480}{-95} = -5$$

(4, –5) is the required solution.

4. FUNCTIONS

4.1 Meaning

A simple technical term in mathematics used to **describe** and **symbolise relationship** between variables is the term "function". The notion of function is essentially very simple as it can be understood easily once a few examples of functions as applied in scientific work, economics and every day life are given. When a gas is subjected to pressure at a constant temperature, its volume **varies with** the pressure (i.e.) volume is a function of the pressure. The inland postage amount paid on a percel **depends on** the weight of the parcel (i.e.) postage expenses on a parcel is a **function** of the weight of the parcel. Finally, in economics, the amount demanded of a commodity by a consumer is in some way connected with the prevailing market price of the commodity (i.e.) demand is a **function** of price. Consumption expediture of a household depends on its income (i.e) household consumption is a function of its income.

In each of these examples, there are two variable quantities which do not change independently of each other; there is a connection between corresponding values, a dependence of one quantity upon the other. Thus, the idea of a **function** involves the conceptof a **relation** between the values of two variables and the **dependence** of one variable on the other.

Space for hints

Check your Progress

5. What do you mean by function?

Space for hints

4.2 Types of Functions

4.2.1 Implicit and Explicit Functions

Based on the nature of dependence of one variable on the other, functions may be classified into two types as **Implicit** function and **Explicit** function. If the relation between two variables is **mutual** (i.e.) if **either variable determines the other**, the function is said to be implicit. Suppose x and y are two variables and also suppose, given the value of x, the value y can take is fixed and similarly conversely. Here x and y are said to have implicit functional relationship. The implicit function relating to two variables is usually represented by the notation.

$f(x, y) = 0$ (to be read as f of x, y is equal to zero)

For example, $2x-3y+5 = 0$ is an implicit function. Suppose we give a value 2 to x, then the value of y will be 3*. Similarly, suppose we give a value 7 to y, then the value of x will be 8. Thus, in an implicit function, given an arbitrary value to either of the two variables, the value of the other variable is fixed.

On the contrary, if the value of one variable (say, x), 'determines' the value of the other variable (say, y), it is called an 'explicit function' and is denoted by $y = f(x)$ (to be read as y is equal to f of x). Here only the variable on the right hand side namely, x can be given an arbitrary value and the value of y is fixed accordingly. For example, $y = 3x+7$ is an explicit function; when x is given an arbitrary value say, 6 the value y takes is fixed as 25.

In the case of an explicit function, the relation is regarded from a definite angle. In our example, it is regarded as y depending on x; here, y is called the dependent variable and x, the **independent** variable. Here any arbitrary value is given only to x and the corresponding value of y is computed accordingly. Similarly, we can also take x as the dependent variable, y the independent variable and write the function as

$$x = f(y)$$

In the case of explicit functions, the variables are arbirtrarily separated and there is no discrimination between the variables except for convenience.

This means that any implicit function can be written as two explicit functions. Taking the same example of implicit function given earlier

---

* Putting $x = 2$, we get $2\times2-3y+5 = 0$ (i.e) $4-3y+5 = 0$ (i.e.) $9-3y = 0$ (i.e.) $3y = 9$ (i.e.) $y = 9/3 = 3$.

namely, $2x-3y+5 = 0$, we can illustrate this point. Keeping x alone on the left hand side and bringing all others to he right hand side, the above function can be rewritten as $x = \frac{3y-5}{2}$ which takes the form $x = f(y)$. Similarly, we can also rewrite it as $y = \frac{2x+5}{3}$ which takes the form of $y = f(x)$. Thus, every implicit function can be written as two explicit functions; one explicit function is considered as the inverse of the other.

Space for hints

An important point to be noted here is that ***a function is not a causal relation.*** That is, one variable does not cause the other. ***Causal relations occur only between quantities of actual phenomena and when such a relation is interpreted by a function, only an explicit function is used and its inverse is ignored.*** That is, when a causual relation is represented by a function only one view of the function is dominant and the other view is neglected.

4.2.2 Linear and Non-linear Functions

Consider the following functions : $y = 3x+5$; $y = x-7$ and $y = 6-x$. All such functions can be included in the single formula, $y = a+bx$ where a and b are any definite numbers, positive, negative or zero. a and b are called 'parameters' and the formula represents the general form of a 'linear function' of x. The term 'linear' is used since as we see later such a function is represented graphically by a straight line. In the linear function, the highest power of each variable is one and no product of the two variables is present in the function. If the highest power of atleast one variable is greater than one and / or an item has product of the two variables, then the graph of the function will not be a straight line but only be a curve and such a function is described as 'non-linear' function. A few examples of 'non-linear' functions are as follows.

$$y = x^2+5x+8$$

$$y = 2x^2+3x-7$$

$$y = x^3+2x^2-x+6$$

$$xy = 5$$

Check your Progress

6. Distinguish between
(i) Implicit and Explicit functions
(ii) Linear and Non-linear functions

4.2.3 Homogeneous Functions

Suppose a function $y = f(x)$ is given. When the independent variable, x is multiplied by any number say, t the right hand side of the function becomes $f(tx)$. Let the corresponding value of y be denoted by y_1.

$$\therefore \quad y_1 = f(tx)$$

Space for hints

If the given function is homogeneous, then it is possible to write the value of y_1 as

$$y_1 = f(tx) = t^n f(x) = t^n y$$

where n is any positive value. n is called the 'degree of homogeneity' and the given function is called 'homogeneous function of degree n'. Let us consider the following example.

Let $$y = ax^5$$

Suppose we double x (i.e) we put 2x in the place of x (in this example, t = 2). The new value of y is

$$y_1 = a(2x)^5 = a2^5x^5 = 2^5 ax^5 = 2^5 y$$

Here $$n = 5$$

∴ The given function is homogeneous of degree 5.

Suppose the given function is,

$$y = a+x^5$$

Here if we put 2x in the place of x,

$$y_1 = a+(2x)^5 = a+2^5 x^5.$$

In this example, **we are not able to express y_1 as a multiple of y as t^n y.**

∴ In this case, the given function is **not a homogeneous function.**

We may come across functions with two or more independent variables also. For simplicity sake let us consider a function with two independent variables say x and y and the dependent variable be denoted by z.

Now, the function is written as

$$z = f(x, y)$$

Here, z is said to be a homogeneous function of degree n, if

$$z_1 = f(tx, ty) = t^n f(x, y) = t^n z.$$

We can extend this to cases of any number of independent variables.

4.2.4 Linear Homogeneous Function :

Check your Progress

7. Define homogeneous function.

When the degree of homogeneity, n = 1, the given function is said to be 'linear homogeneous'. For example, consider the function,

$$z = ax + by$$

Put tx in the place of x and ty in the place of y.

Now, $z_1 = a(tx)+b(ty)$

$= t(ax+by) = tz.$

∴ We can give the definition of linear homogeneous function as follows.

A function is said to be linear homogeneous if the dependent variable increases in the same proportion in which the independent variables are increased simultaneously. That is, when the values of independent variables are doubled simultaneously, the dependent variable also gets doubled; when the independent variables are trebled simultaneously, the dependent variable is also trebled; and so on.

Example :

Find whether the following function is a homogeneous function or not. If it is homogeneous, what is the degree of homogeneity?

$Q = AK^a L^b$ where Q, K and L are variable and others are constants.

Answer

Given function is $Q = AK^a L^b$

Replacing K by tK and L by tL on the RHS of the function, we get

$$A(tK)^a (tL)^b = At^aK^a t^bL^b$$
$$= t^at^b AK^aL^b$$
$$= t^{(a+b)} Q = t^nQ \text{ (putting } n = a+b)$$

∴ Given function is a homogeneous function and the degree of homogeneity is (a+b).

Example

Verify whether the following function is alinear homogeneous function.

$$Q = AK^a L^{1-a}$$

Answer

Suppose K and L in the given function are multiplied by t and the resultant value of the function is denoted by Q_1. Now,

$$Q_1 = A(tK)^a (tL)^{1-a}$$
$$= At^aK^a t^{1-a} L^{1-a}$$
$$= t^a . t^{1-a} AK^a L^{1-a} = t^{a+1-a} . Q = tQ$$

∴ Given function is a linear homogeneous function.

Space for hints

Check your Progress

8. What do you understand by linear homogeneous function?

Space for hints

5. Functions and Curves

Whenever we are given a function, it is always possible to construct a table of corresponding values of the two variables x and y by giving definite values to one variable (say, x) and solving the formula of the function for the correspondign values of the other variable, y in each case. Each pair of values may be plotted as points on a graph paper and a freehand curve may be drawn through them. This freehand curve is called 'graph' of the function. To each given function relating variables x and y, there corresponds a set of points comprising a curve; the analytical property defined by the function is reflected in the geometrical property common to all points on the curve. The converse is also true. A curve is simply a collection of points in a plane with a common characteristic and this common characteristic can be translated into an analytical relation between x and y; the analytical relation established is called the 'equation of the curve'.

Straight line graphs, U-shaped curves and inverted U-shaped curves are the commonly used curves in Economics. The equations of these curves are as follows.

Curve	Equation
Straight line	$y = a + bx$
U-shaped curve	$y = ax^2$
Inverted U-shaped curve	$y = -ax^2$

6. Equation of a straight line

The simplest curve class is the class of straight lines. A straight line is fixed if two points on it are specified. We can, therefore, speak of a straight line PQ, where P and Q are two fixed points on the line. However, we must always remember that a striaght line can be extended to any length in both directions. This fact is illustrated below :

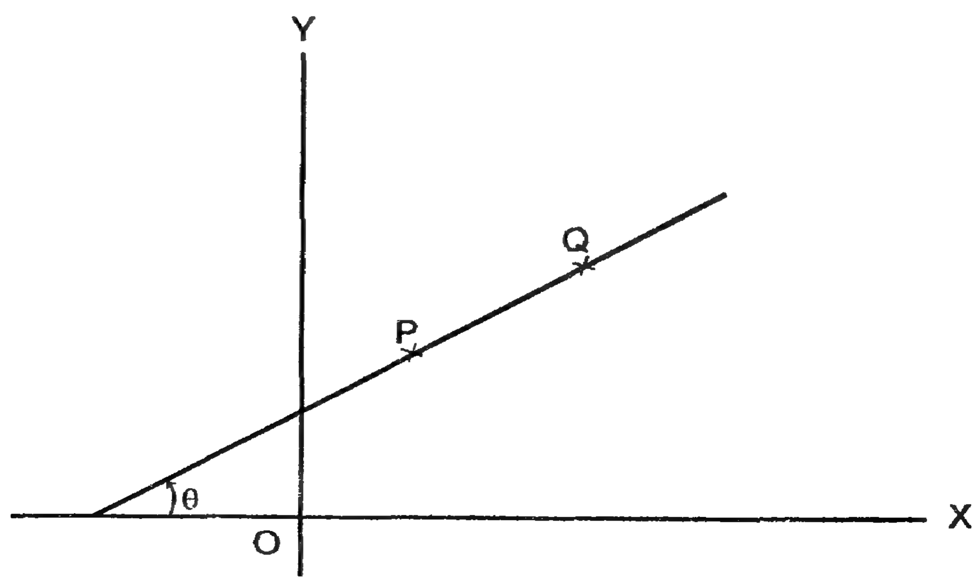

Fig. (1)

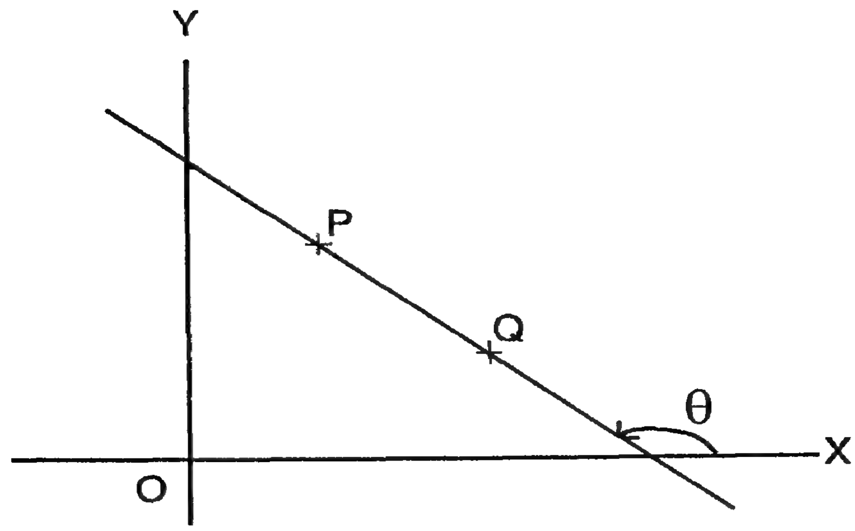

Fig. (2)

6.1 Slope - Meaning and Measurement

An important property of a given straight line is its **direction** relative to the fixed axes OX and OY. The direction of a line is defined by means of an angle q that the line makes with the positive direction of X axis, namely, OX. If the angle q is acute (that is, less than 90°) the straight line is said to slope upwards from left to right as shown in Fig(1) ; if q is obtuse (that is, greater than 90°) the straight line is said to slope downwards from left to right as shown in Fig (2). The angular measure of direction is not convenint for most analytical purposes. We need an indicator in terms of lengths to denote the direction of a line. Such an indicator is provided by the concept of **slope** or **gradient**.

Slope of a straight line is defined as tan q where q is angle made by the striaght line with the positive direction of the x-axis in the anti-clockwise direction. We have illustrated the computation of slope with the help of a diagram as follows :

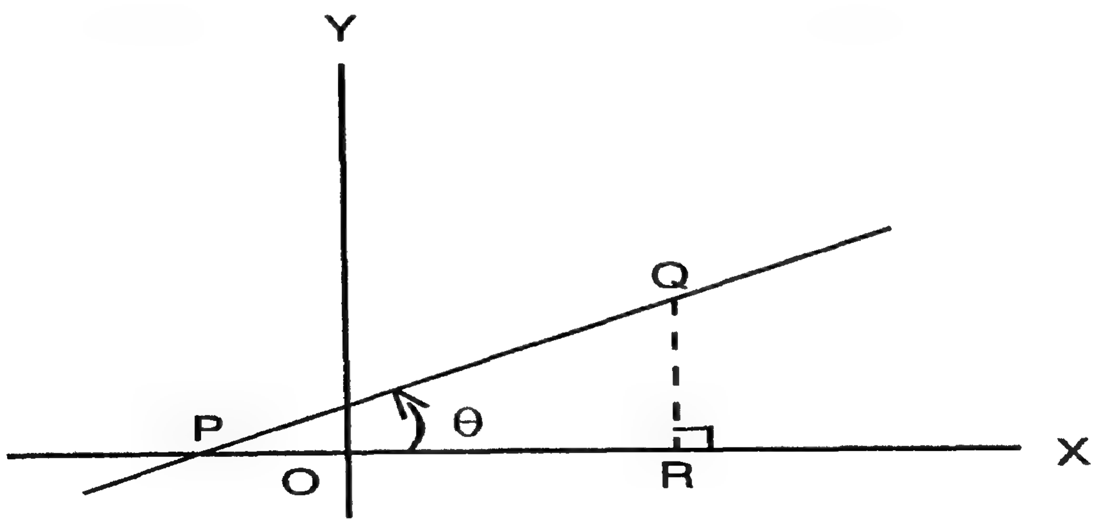

Fig. (3)

Straight line PQ makes an angle q with the positive direction of x-axis. Now,

$$\text{Slope of PQ} = \tan q = \frac{QR}{PR}$$

Space for hints

Some of the important points relating to the concept of slope are as follows :

(i) The slope of x-axis or lines parallel to x-axis $= 0$.

(ii) The slope of y-axis or lines parallel to y-axis $= \infty$

(iii) All parallel lines have the same slope.

(iv) A line sloping upwards has a positive slope and a line sloping downwards has a negative slope.

(v) The steeper the line, larger is the numerical value of its slope.

(vi) The slope is usually denoted by the letter m.

6.2 Types of equation to a straight Line

The equation of a straight line may be given in any one of the following five different forms :

(i) Slope-intercept form

(ii) Point-slope form

(iii) Two points form

(iv) Intercepts form

(v) General form.

(i) Slope - intercept form of equation :

In this form, the equation is given as,

$$y = mx + c$$

where m denotes the slope of the line and c denotes the intercept on the y-axis.

If the straight line cuts the Y axis at a point say, B, then OB is called the intercept on the Y axis. That is, $c = OB$.

The angle and the intercept OB for different kinds of lines are shown in the following diagrams.

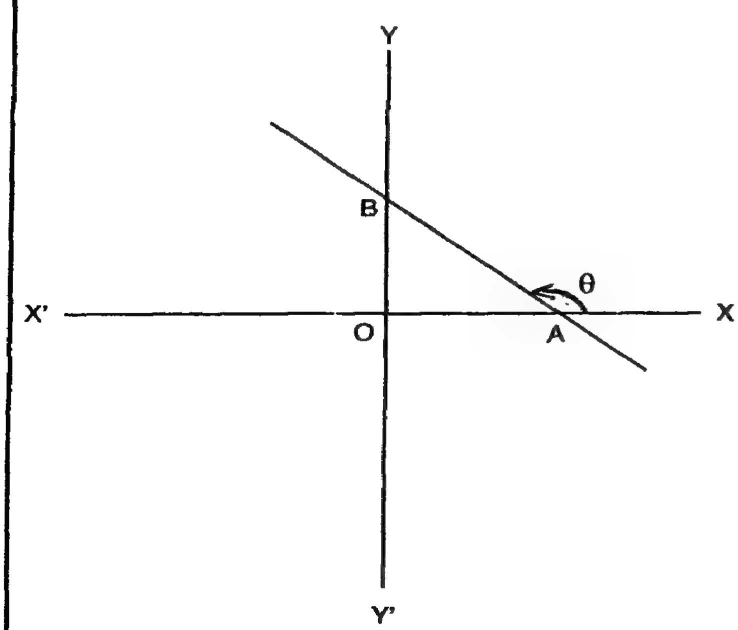

Fig. (4)

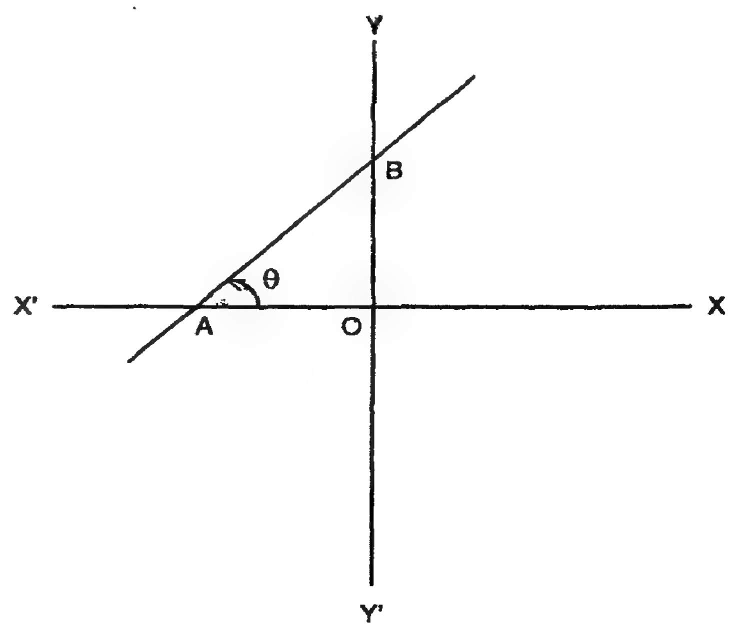

Fig. (5)

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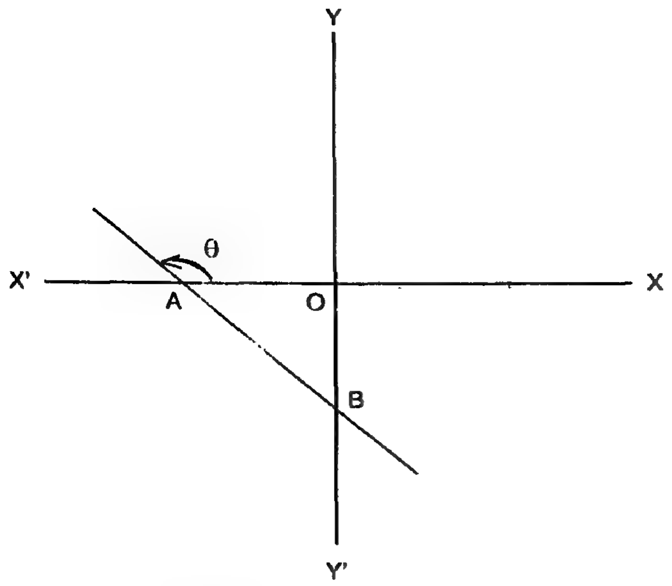


Fig. (6)

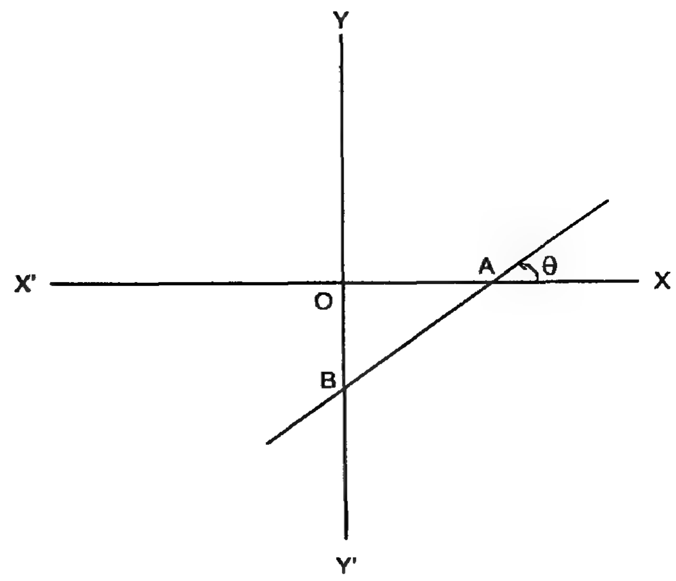


Fig. (7)

Example :

Find the equation to the straight line given that $m = \frac{1}{2}$ and $c = 3$.

Equation to the straight line is $y = mx + c$.

It is given that $m = \frac{1}{2}$ and $c = 3$

Putting these values in the equation, we get

$$y = \tfrac{1}{2} x + 3$$

(i.e.) $2y = x + 6$

(i.e.) $x - 2y + 6 = 0$.

Example :

Find the equation to the straight line if $q = 60°$ and $c = 4$.

Equation to the straight line is

$$y = mx + c$$

When $q = 60°$, $\tan q = \tan 60° = \sqrt{3}$

$\therefore\ m = \sqrt{3}$

Given, $c = 4$

Putting these values in the equation to the straight line, we get

$$y = \sqrt{3}\,x + 4$$

(ii) Equation of a straight line in point-slope form :

Suppose a straight line passes through a point whose coordinates are (x_1, y_1). Also suppose that the slope of the line is m. Now, the equation of the straight line is given as,

$$(y - y_1) = m\,(x - x_1)$$

Space for hints

Example 1:

Find the equation of the line passing through $(-7,8)$ and having an inclination of $135°$.

Equation of the line having slope m and passing through (x_1, y_1) is $(y - y_1) = m(x - x_1)$.

$m = \tan 135° = \tan(180° - 45°) = -\tan 45° = -1$

(x_1, y_1) is $(-7,8)$ (i.e.) $x_1 = -7$, $y_1 = 8$

$\therefore\ y - 8 = (-1)[x-(-7)]$

$y - 8 = -(x + 7)$

$y - 8 = -x - 7$

$x + y - 8 + 7 = 0$

$x + y - 1 = 0$

Example 2:

A line with slope (-2) passes through the point $(4,-1)$. **Find the** equation of the line $m = -2$; (x_1, y_1) is $(4, -1)$

$\therefore$ Equation of the line is $y - y_1 = m(x - x_1)$

$y - (-1) = -2(x - 4)$

$y + 1 = -2x + 8$

$2x + y + 1 - 8 = 0$

$2x + y - 7 = 0$

Example 3:

Find the equation of the line whose slope is $\sqrt{3}$ and passes through $(2, 3)$.

Equation of the straight line with slope m and passing through (x_1, y_1) is

$(y - y_1) = m(x - x_1)$

Given, $m = \sqrt{3}$ and $x_1 = 2$, $y_1 = 3$.

$\therefore$ The equation becomes

$(y - 3) = \sqrt{3}(x - 2)$

Example 4:

Write down the equation of the straight line passing through $(4, -5)$ with slope equal to $-2/3$.

Space for hints

Equation of the straight line is,

$$(y - y_1) = m (x - x_1).$$

Given, $m = -2/3$, $x_1 = 4$, $y_1 = -5$

$$\therefore [y - (-5)] = \frac{-2}{3} (x - 4)$$

$$y + 5 = \frac{-2}{3} (x - 4)$$

$$3 (y + 5) = -2 (x - 4)$$

$$3y + 15 = -2x + 8$$

$$\therefore 2x + 3y + 15 - 8 = 0$$

$$2x + 3y + 7 = 0.$$

(iii) Equation of a straight line in two points form :

Suppose a straight line passes through the two points (x_1, y_1) and (x_2, y_2). Now, the equation of the line is,

$$\frac{y - y_1}{y_1 - y_2} = \frac{x - x_1}{x_1 - x_2}$$

Example 1:

Find the equation of the straight line joining the points (3, 2) and (5, 0).

Let (x_1, y_1) be (3,2) and (x_2, y_2) be (5,0)

$\therefore$ $x_1 = 3$; $x_2 = 5$

$y_1 = 2$, $y_2 = 0$

Equation to the given line is,

$$\frac{y - y_1}{y_1 - y_2} = \frac{x - x_1}{x_1 - x_2}$$

$$\frac{y - 2}{2 - 0} = \frac{x - 3}{3 - 5}$$

$$\frac{y - 2}{2} = \frac{x - 3}{-2}$$

$$-2 (y - 2) = 2(x - 3)$$

$$-2y + 4 = 2x - 6$$

$$-2x - 2y + 4 + 6 = 0$$

$$-2x - 2y + 10 = 0$$

$$x + y - 5 = 0$$

Space for hints

Example 2:

Find the equation of the straight line joining the points (2, 3) and (3, 4)

Equation of the straight line joining the points (x_1, y_1) and (x_2, y_2) is,

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2}$$

Given, $(x_1, y_1) = (2, 3)$

$(x_2, y_2) = (3, 4)$

$\therefore$ The equation becomes

$$\frac{x-2}{2-3} = \frac{y-3}{3-4}$$

$$\frac{x-2}{-1} = \frac{y-3}{-1}$$

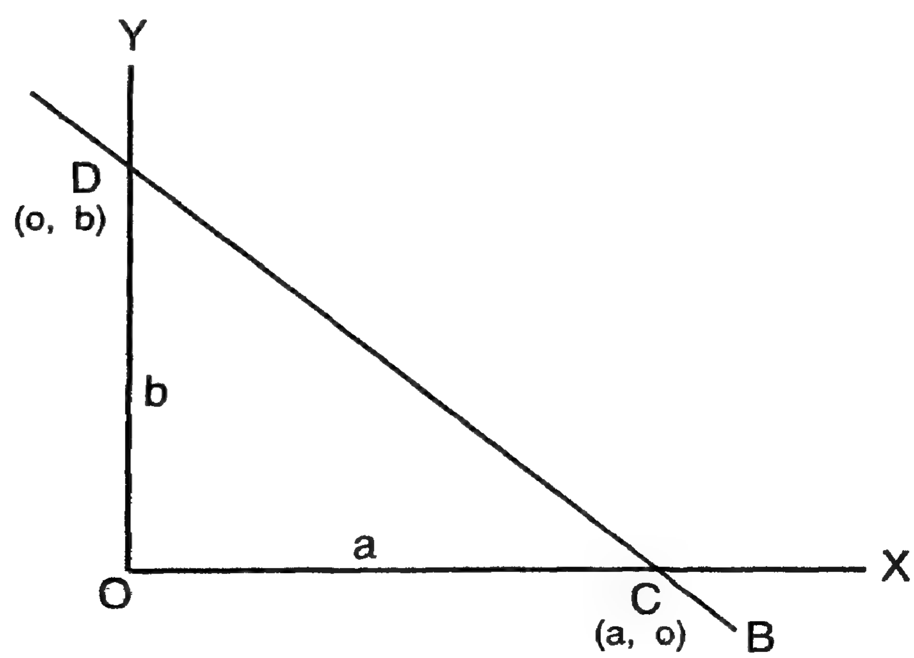
$$x - 2 = y - 3$$

$$x - y - 2 + 3 = 0$$

$$x - y + 1 = 0.$$

(iv) Equation of a straight line in intercepts form:

Let a straight line make an intercept of a on the X axis and b on the Y axis as shown in the following diagram.



Now the equation of the straight line is given as,

$$\frac{x}{a} + \frac{y}{b} = 1$$

Example 1:

Space for hints

Write the equations of the lines given the x intercept a and the y intercept b, as follows :

i) $a = 3, b = 2$ (ii) $a = -3/2$, $b = 4/3$ (iii) $a = h$, $b = k$;

(i) $a = 3$, $b = 2$.

Equation to the straight line is

$$\frac{x}{a} + \frac{y}{b} = 1$$

$$\frac{x}{3} + \frac{y}{2} = 1, \quad \frac{2x + 3y}{6} = 1$$

$$2x + 3y = 6$$

ii) $a = \frac{-3}{2}$ $\quad b = \frac{4}{3}$

Equation to the line is $\frac{x}{a} + \frac{y}{b} = 1$

$$\frac{x}{\frac{-3}{2}} + \frac{y}{\frac{4}{3}} = 1$$

$$\frac{-2x}{3} + \frac{3y}{4} = 1$$

$$\frac{-8x + 9y}{12} = 1$$

$$-8x + 9y = 12$$

$$-8x + 9y - 12 = 0$$

$$8x - 9y + 12 = 0$$

(ii) $a = h$, $\quad b = k$

Equ. to the line : $\frac{x}{h} + \frac{y}{k} = 1$

Note :

1. Equation to the X axis is $y = 0$.
2. Equation to the Y axis is $x = 0$.
3. Equation to line parallel to x-axis making an intercept c on the Y axis is $y = c$.
4. Equation to line parallel to y-axis making an intercept k on the X axis is $x = k$.

Space for hints

5. If any point is said to be on a given straight line, it meant that the co-ordinates of the point satisfy the equation of the given line. For example, (x_1, y_1) be the given point. In the equation of the straight line, when x,y are replaced by x_1, y_1 respectively, if the R.H.S = L.H.S of the equation, the point is said to satisfy the equation.

Example 2:

The equation of a straight line is $2x + y - 7 = 0$. Find whether the point $(4, -1)$ lies on this line.

The equation of the straight line is $2x + y - 7 = 0$.

Put $x = 4$ and $y = -1$ in the L.H.S of the equation.

$2 \times 4 - 1 - 7 = 8 - 8 = 0$ = R.H.S of the equation.

$\therefore$ The point $(4, -1)$ lies on the given line.

Example 3:

A straight line makes the intercepts 2 on the x-axis and 3 on the y-axis and passes through a point with x co-ordinate 4. Find the y co-ordinate of the point.

Intercept form of the equaiton of straight line is.

$$\frac{x}{a} + \frac{y}{b} = 1$$

It is given that $a = 2$ and $b = 3$.

$\therefore$ The equation becomes

$$\frac{x}{2} + \frac{y}{3} = 1$$

(i.e.) $\dfrac{3x + 2y}{6} = 1$

(i.e.) $3x + 2y = 6$

(i.e.) $3x + 2y - 6 = 0$

This line passes through a point with x co-ordinate equal to 4.

$\therefore$ It must satisfy the equation of the line. That is,

$3 \times 4 + 2y - 6 = 0.$

$12 + 2y - 6 = 0$

$6 + 2y = 0$

Space for hints

$$2y = -6$$

$$\therefore \ y = \frac{-6}{2} = -3$$

y co-ordinate of the point is –3.

(v) General form of the equation of a straight line :

The general form is given as,

$$ax + by + c = 0.$$

Here the slope of the line and the x and y intercepts are given as follows.

$$\text{Slope} = \frac{-a}{b} = \frac{-\text{coefficient of x}}{\text{coefficient of y}}$$

$$\text{x intercept} = \frac{-c}{a} = \frac{-\text{constant term}}{\text{coefficient of x}}$$

$$\text{y intercept} = \frac{-c}{b} = \frac{-\text{constant term}}{\text{coefficient of y}}$$

Example :

Find the slope and x,y intercepts of the line $x - 2y + 6 = 0$.

Equation of the given line is $x - 2y + 6 = 0$.

The general form of the equation of a line is,

$$ax + by + c = 0$$

comparing this equation with the given equation, we get,

$a = 1$, $b = -2$ and $c = 6$.

$$\therefore \text{Slope} = \frac{-a}{b} = \frac{-1}{-2} = \frac{1}{2}$$

$$\text{X intercept} = \frac{-c}{a} = \frac{-6}{1} = -6$$

$$\text{Y intercept} = \frac{-c}{b} = \frac{-6}{-2} = +3$$

7. Answers to Check Your Progress Questions :

1. Refer 1.1
2. Refer 1.5
3. Refer 2.1
4. Refer 3.2.1
5. Refer 4.1
6. Refer 4.2.1 & 4.2.2
7. Refer 4.2.3
8. Refer 4.2.4

Space for hints

8. Model questions for guidance :

10 Marks Questions (One Page Answer)

1. a) Define a function.

   b) Solve

   $$9x - 3y = 1$$
   $$5x + 4y = 14$$

2. a) Define a function and give an example.

   b) Solve

   $$4x - 3y = 7$$
   $$3x + 2y = 9$$

3. Solve

   (a) $3x^2 + 10x + 8 = 0$

   (b) $3x - 2y = 13$

   (c) $5x + 3y = 66$

4. Define :

   (a) variables

   (b) functions

   (c) Equations

5. Solve the following equations :

   $$x + \frac{4}{y} = 3$$
   $$y + \frac{4}{x} = -3$$

6. Solve the following pairs of simultaneous equation :

   (a) $3x + 2y = 13$
   $2x - 3y = 12$

   (b) $4x - 3y - 15 = 0$
   $3x - 3y - 6 = 0$

7. Explain linear homogeneous functions with an example.

20 Marks Questions (Three Page Answer)

1. State and explain the important properties of linear homogeneous function.

2. Using the general formula, solve the equations

   $3x + y = 1$ and $5x + 2y = 3$

Space for hints

UNIT – 9

MATRICES

Introduction :

An arrangement of numbers in rows and columns given within brackets is called 'matrix' in mathematics. Just like as in the case of ordinary numbers various operations like addition, subtraction, multiplication are done with matrices. Matrix inversion is an operation which is useful in solving simultaneous linear equations. Determinant is another mathematical concept related to square matrix where number of rows and number of columns are equal. Determinant is also useful in solving simultaneous linear equations. In this Unit-9, we describe the meaning and application of matrices and determinants.

Unit Objectives :

After studying this Unit, you would be able to understand

* the meaning, types and operations with matrix
* the method of solving equations with matrix
* the meaning and evaluation of a determinants
* the Cramer's rule in solving equations.

Unit Structure

1.Matrix

2 Determinants

3. Solving Simultaneous Linear Equations

4. Answers to Check Your Progress Questions

5. Model questions for guidance :

1.Matrix

Definition :

A rectangular arrangement of mn numbers into m rows and n columns is called a matrix of order m x n (read as m by n matrix or m into n matrix). A matrix is simply a convenient way of representing arrays of numbers.

Each of the mn numbers of the matrix is called an element of the matrix. They are also called scalars.

The mn elements of the matrix need not all be different. Some or all of them may be equal.

Space for hints

The elements constituting a matrix are usually enclosed in one of the following three different types of brackets [ ], ( ), { } . However, [ ] is the most commonly used bracket.

Examples

(i) $\begin{bmatrix} 2 & 3 & 6 & -1 \\ 8 & 1 & 4 & 3 \end{bmatrix}$ is a 2 x 4 matrix because it contains 2 rows and 4 columns.

(ii) $\begin{bmatrix} 4 & 8 \\ -9 & 0 \\ 6 & 5 \\ 0 & 4 \\ 8 & 3 \end{bmatrix}$ is a 5 x 2 matrix.

(iii) $\begin{bmatrix} 68 & 11 & 44 & 0 \\ 7 & 6 & 104 & -11 \\ 0 & 129 & 49 & 68 \end{bmatrix}$ is a 3 x 4 matrix.

(iv) [0 25 3 40 –5] is a 1 x 5 matrix and is also called a row matrix or row vector.

(v) $\begin{bmatrix} 0 \\ 8 \\ 7 \\ 6 \end{bmatrix}$ is a 4 x 1 matrix and is called column matrix or column vector.

(vi) $\begin{bmatrix} 4 & 8 & -3 \\ 0 & 1 & 2 \\ 1 & 0 & 5 \end{bmatrix}$ is a matrix of order 3 x 3.

1.2 General Notation of a matrix :

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & & & & \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

A is a matrix of order m x n. This is also written in an abbreviated form as $A = [a_{ij}]$; i = 1, 2 .... m ; j = 1, 2 .... n. It is often referred to as the m x n matrix $[a_{ij}]$ or $[a_{ij}]_{m \times n}$.

In this matrix A, the rows are numbered from top to bottom and the columns are numbered from left to right. The element a_{ij} is called the general element or the element in the i th row and j th column or (i, j)th element of matrix ; i-and j are called suffixes. The first suffix i, will always denote rows and the second suffix j, will always denote the columns. Thus, all elements in the second row have 2 as first suffix and all the elements in the fifth column have 5 as the second suffix.

Check your Progress

1. Define Matrix.

The first members of each row constitute the first column ; the second members of each row constitute the second column and, in general, the j th members of each row constitute the j th column.

Space for hints

The m x n matrix has therefore m rows and n columns and the suffixes i and j respectively range from 1 to m and 1 to n.

The first suffix is invariant for each row and the second suffix is invariant for each column.

The elements a_{ij} of any matrix $[a_{ij}]$ for which i = j are called the diagonal elements and the line along which they lie is called the Principal diagonal or usually called the diagonal of the matrix.

Note : Matrices are represented by capital letters A, B, C, D, P, Q, etc. Their (i, j)th elements will be denoted by a_{ij}, b_{ij}, c_{ij}, d_{ij}, p_{ij}, q_{ij} etc., respectively.

Example : 1

Read off the elements a_{11}, a_{21}, a_{32} and a_{54} in each of the following matrices whenever these elements exist.

a) $\begin{bmatrix} 0 & 2 & 3 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$

c) $\begin{bmatrix} 5 & 6 \\ 7 & -3 \\ 8 & 7 \end{bmatrix}$ d) $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 3 & 2 \\ 6 & 3 & 2 & 1 \\ 4 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

a) a_{11} is the first row element in the first column of $[0 \quad 2 \quad 3]$

Therefore $a_{11} = 0$

Similarly a_{21} is the element in the second row and 1st column of [0, 2, 3]. Since the given matrix has only 1 row and the second row does not exist, a_{21} does not exist. Similarly, a_{32}, a_{54} do not exist for this matrix.

b) a_{11} is the element in the 1st row and 1st column of $\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$

Therefore $a_{11} = 1$.

Similarly, $a_{21} = 4$.

However, a_{32} and a_{54} do not exist.

c) $a_{11} = 5$, $a_{21} = 7$, $a_{32} = 7$, a_{54} does not exist.

d) $a_{11} = 1$, $a_{21} = 0$, $a_{32} = 3$, $a_{54} = 1$.

Space for hints

Example : 2

Read off the second row and third column of the following matrices :

a) $\begin{bmatrix} 1 & 5 & 4 \\ 2 & 3 & 6 \end{bmatrix}$ b) $\begin{bmatrix} 7 & 8 & 9 & 1 & 0 \end{bmatrix}$

c) $\begin{bmatrix} 2 & 3 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$ d) $\begin{bmatrix} 2 & 3 & 7 & 2 \\ ½ & 3 & 2 & 3 \\ 0 & 0 & 2 & 3 \end{bmatrix}$

a) The second row of $\begin{bmatrix} 1 & 5 & 4 \\ 2 & 3 & 6 \end{bmatrix}$ is given by $\begin{bmatrix} 2 & 3 & 6 \end{bmatrix}$

The third column of this matrix is given by $\begin{bmatrix} 4 \\ 6 \end{bmatrix}$

b) [7 8 9 1 0] has no second row,

The third column is given by [9]

c) The second row of $\begin{bmatrix} 2 & 3 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$ is given by $\begin{bmatrix} 1 & 1 \end{bmatrix}$

There is no third column for this matrix.

d) The second row is given by [½ 3 2 3]

The third column is given by $\begin{bmatrix} 7 \\ 2 \\ 2 \end{bmatrix}$

Example : 3

Write down the diagonal elements of $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 4 & 6 & 7 \end{bmatrix}$

The given matrix is a 3 x 3 matrix.

∴ The diagonal elements are

$a_{11} = 1$; $a_{22} = 2$; and $a_{33} = 7$.

1.3 Types of Matrices

(i) Null Matrix :

A matrix in which all the elements are Zeros is called as a Null Matrix.

Example :

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{2x3} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}_{4x2} \quad C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3x3}$$

(ii) Square Matrix :

When the number of rows in a matrix is equal to the number of columns in a matrix (m = n), it is called a square matrix.

Example :

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}_{2x2} \qquad B = \begin{bmatrix} 5 & 7 & 2 \\ 1 & 3 & 0 \\ 1 & 5 & 6 \end{bmatrix}_{3x3}$$

Rows = 2 Rows = 3

Columns = 2 Columns = 3

(iii) Diagonal Matrix :

A Diagonal matrix is a square matrix where all elements except the main diagonal elements are zeros. Main diagonal of a square matrix is composed of all elements aij of the matrix for which i = j

Example :

(i) $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

i = j

A is a diagonal matrix of order 2

$a_{11} = 2; a_{22} = 1$

(ii) $B = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 9 \end{bmatrix}_{3x3}$

$a_{11} = 5$

$a_{22} = 7$

$a_{33} = 9$

B is a diagonal matrix of order 3.

(iv) Scalar Matrix :

A scalar matrix is a diagonal matrix where all the main diagonal elements are equal.

Example :

(i) $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}_{2x2}$

$a_{11} = 2$ $a_{22} = 2$ $a_{11} = a_{22}$

Space for hints

Check your Progress

2. What is Null Matrix?
3. What is Square Matrix?
4. What is Diagonal Matrix?
5. What is Scalar Matrix?

Space for hints

(ii) $B = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}_{3x3}$

$a_{11} = 9 \quad a_{22} = 9 \qquad a_{33} = 9$

i.e., $a_{11} = a_{22} = a_{33}$

(v) Triangular matrix :

A square matrix $A = [a_{ij}]$ in which $a_{ij} = 0$ for $i > j$ is called an upper triangular matrix. In $A = [a_{ij}]$ if $a_{ij} = 0$ for $i < j$, A is called a lower triangular matrix. The elements below the main diagonal of an upper triangular matrix are zero.

Example :

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}; \begin{bmatrix} 1 & 2 & 5 \\ 0 & -3 & 6 \\ 0 & 0 & 6 \end{bmatrix}; \begin{bmatrix} 5 & -1 & 7 & 9 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

are all upper triangular matrices.

$$\begin{bmatrix} 3 & 0 \\ 8 & 1 \end{bmatrix}; \begin{bmatrix} 4 & 0 & 0 \\ 5 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix}; \begin{bmatrix} 6 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 5 & -3 & 6 & 3 \end{bmatrix}$$

are all lower triangular matrices.

Here the elements above the main diagonal are zero.

Note : The diagonal matrix $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and

The scalar matrix $\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ are upper triangular as well as lower triangular.

(vi) Unit Matrix or Identity Matrix :

An identity matrix is a diagonal matrix, the diagonal elements being equal to unity or one. It is always denoted by the letter I.

Example :

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2x2} \qquad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3x3}$$

(vii) Transpose of a Matrix :

When the rows and columns of a given matrix are interchanged, we get a new matrix which is called the transpose of the given matrix. If A is the given matrix, the transpose of A is denoted by A^T or A'.

Example :

(i) $A = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 7 & 1 \end{bmatrix}_{2x3}$ $\quad A^T$ or $A' = \begin{bmatrix} 2 & 5 \\ 3 & 7 \\ 4 & 1 \end{bmatrix}_{3x2}$

(ii) $A = \begin{bmatrix} 5 & 7 & 9 \\ 0 & 2 & 1 \\ 4 & 6 & 8 \end{bmatrix}_{3x3}$ $\quad A^T = \begin{bmatrix} 5 & 0 & 4 \\ 7 & 2 & 6 \\ 9 & 1 & 8 \end{bmatrix}_{3x3}$

(viii) Symmetric Matrix :

A square matrix in which $a_{ij} = a_{ji}$ for i ¹ j is called a symmetric matrix. That is, if $A = A^T$, it is called a symmetric matrix.

Example :

(i)

$A = \begin{bmatrix} 2 & 5 \\ 5 & 7 \end{bmatrix}$ $\quad a_{12} = a_{21} = 5.$ (i.e.,) $A = A^T$

$\therefore$ A is a symmetric matrix.

(ii) $B = \begin{bmatrix} 2 & 8 & 7 \\ 8 & 5 & 3 \\ 7 & 3 & 6 \end{bmatrix}$ $\quad \begin{matrix} a_{12} = a_{21} = 8 \\ a_{13} = a_{31} = 7 \\ a_{23} = a_{32} = 3 \end{matrix}$

(i.e.,) $B = B^T$. $\therefore$ B is a symmetric matrix.

(ix) Skew Symmetric Matrix :

A square matrix in which $a_{ij} = (-a_{ji})$ is called a skew symmetric matrix.

Example :

(i) $A = \begin{bmatrix} 8 & -5 \\ 5 & 9 \end{bmatrix}$

$a_{12} = -5$ $\qquad a_{21} = 5$

That is, $a_{12} = -a_{21}$ $\qquad \therefore$ A is skew symmetric.

(ii) $A = \begin{bmatrix} 7 & -5 & 3 \\ 5 & 8 & -7 \\ -3 & 7 & 2 \end{bmatrix}$

$a_{12} = -a_{21} = -5$ $\qquad a_{23} = -a_{32} = -7$ $\qquad a_{13} = -a_{31} = 3$

(x) Equal Matrices :

Two matrices of the same order are said to be equal only if all the corresponding elements in the two matrices are equal.

Space for hints

Check your Progress

6. Define Unit Matrix?
7. What is Transpose of a Matrix?
8. What is meant by Symmetric Matrix?
9. What is meant by Equal Matrix?

Space for hints

Example :

(i) $A = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 8 & 9 \end{bmatrix}_{2x3}$ $\qquad B = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 8 & 9 \end{bmatrix}_{2x3}$

$$A = B$$

(ii) $A = \begin{bmatrix} -1 & 2 & 6 \\ 3 & -1 & 7 \\ 0 & -2 & 1 \end{bmatrix}_{3x3}$ $\qquad B = \begin{bmatrix} -1 & 2 & 6 \\ 3 & -1 & 7 \\ 0 & -2 & 1 \end{bmatrix}_{3x3}$

$$A = B$$

(iii) $A = \begin{bmatrix} 5 & 7 \\ 7 & 2 \\ 1 & 6 \end{bmatrix}_{3x2}$ $\qquad B = \begin{bmatrix} 5 & 7 \\ 1 & 2 \\ 1 & 6 \end{bmatrix}_{3x2}$

$a_{21} = 7$ in A

$b_{21} = 1$ in B. That is, corresponding elements in A and B are not equal.

$\therefore A \neq B$

(xi) Sub Matrices :

Let 'A' be a given matrix. By deleting a few rows and columns, we get a new matrix called the sub-matrices of A.

Example :

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

Sub matrices are,

$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix}$

$[1 \quad 2 \quad 3]$, $[4 \quad 5 \quad 6]$, $[1 \quad 2]$, $[2 \quad 3]$, $[1 \quad 3]$

$[4 \quad 5]$, $[4 \quad 6]$, $[5 \quad 6]$, $[1]$, $[2]$, $[3]$, $[4]$, $[5]$, $[6]$

(xii) Row Matrix :

Row matrix is a matrix with only one row.

Example :

$A = [1 \quad 1]_{1x2}$ $\qquad B = [1 \quad 7 \quad 5 \quad 4 \quad 6]_{1x5}$

(xiii) Column Matrix :

Column matrix is a matrix with only one column.

Space for hints

Example :

(i) $A = \begin{vmatrix} 1 \\ 7 \\ 6 \end{vmatrix}_{3x1}$ (ii) $B = \begin{vmatrix} 7 \\ 8 \\ 6 \\ 5 \end{vmatrix}_{4x1}$ (iii) $C = \begin{vmatrix} 7 \\ 1 \\ 4 \\ 5 \\ 6 \\ 3 \end{vmatrix}_{6x1}$

1.4 Basic Operations

The basic operations in the matrix theory are :

(a) Addition and subtraction of matrices

(b) Scalar multiplication of a matrix

(c) Multiplication of matrices.

1.4.1 Addition of Matrices :

Let $A = [a_{ij}]$ and $B\ [b_{ij}]$ be two matrices of **the same order m x n**. The matrix $C = [c_{ij}]$ where $c_{ij} = a_{ij} + b_{ij}$ is called the sum of the matrices A and B and is written as A + B. Thus, each element of A + B is the sum of the corresponding elements of A and B.

$$\text{Let } A = \begin{bmatrix} 4 & 2 & 0 & -1 \\ 8 & 3 & -2 & 5 \\ 0 & 1 & 7 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & -2 & 0 & 1 \\ -7 & -2 & 2 & -4 \\ 0 & -1 & -6 & 0 \end{bmatrix}$$

Both the matrices are of the same order, 3 x 4.

$$A + B = \begin{bmatrix} 4+2 & 2+(-2) & 0+0 & -1+1 \\ 8+(-7) & 3+(-2) & -2+2 & 5+(-4) \\ 0+0 & 1+(-1) & 7+(-6) & 0+0 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 2-2 & 0 & 0 \\ 8-7 & 3-2 & 0 & 5-4 \\ 0 & 1-1 & 7-6 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

1.4.2 Difference of two matrices

The difference A–B of two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ of the same order mxn is the matrix $D = [d_{ij}]$ where $d_{ij} = a_{ij} - b_{ij}$

Thus, Each element of A–B is the difference of the corresponding elements of A and B.

Space for hints

Let $A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 4 & 0 & 2 & 1 \\ 2 & -5 & 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -4 & 1 & 2 \\ 1 & 5 & 0 & 3 \\ 2 & -2 & 3 & -1 \end{bmatrix}$

Both the matrices are of the same order, 3x4.

$$A - B = \begin{bmatrix} 1-3 & 2-(-4) & -1-1 & 0-2 \\ 4-1 & 0-5 & 2-0 & 1-3 \\ 2-2 & -5-(-2) & 1-3 & 2-(-1) \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 2+4 & -2 & -2 \\ 3 & -5 & 2 & -2 \\ 0 & -5+2 & -2 & 2+1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 6 & -2 & -2 \\ 3 & -5 & 2 & -2 \\ 0 & -3 & -2 & 3 \end{bmatrix}$$

Note : From the definitions it is obvious that A + B and A – B can be obtained only when A and B are of the same order. Further' A + B and A – B are of the same order as A and B.

Example :

If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 \\ 1 & 5 \\ 4 & 3 \end{bmatrix}$

find A + B and A – B,

$$A + B = \begin{bmatrix} 1+3 & 2+2 \\ 3+1 & 4+5 \\ 5+4 & 6+3 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 4 & 9 \\ 9 & 9 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 1-3 & 2-2 \\ 3-1 & 4-5 \\ 5-4 & 6-3 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 2 & -1 \\ 1 & 3 \end{bmatrix}$$

Example :

If $A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix}$

find A + B and A – B.

$$A + B = \begin{bmatrix} 1+(-2) & 3+4 \\ 2+3 & 5+(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 1-2 & 7 \\ 5 & 5-1 \end{bmatrix} = \begin{bmatrix} -1 & 7 \\ 5 & 4 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 1-(-2) & 3-4 \\ 2-3 & 5-(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 1+2 & -1 \\ -1 & 5+1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -1 & 6 \end{bmatrix}$$

Example :

$$A = \begin{bmatrix} 4 & -1 & 0 \\ -3 & 5 & -6 \\ 2 & -7 & 8 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 0 & 1 \\ 5 & -2 & 2 \\ 3 & 4 & 3 \end{bmatrix}$$

Find $A+B$; $A-B$ and $B-A$

$$A+B = \begin{bmatrix} 4+(-1) & -1+0 & 0+1 \\ -3+5 & 5+(-2) & -6+2 \\ 2+3 & -7+4 & 8+3 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ 2 & 3 & -4 \\ 5 & -3 & 11 \end{bmatrix}$$

$$A-B = \begin{bmatrix} 4-(-1) & -1-0 & 0-1 \\ -3-5 & 5-(-2) & -6-2 \\ 2-3 & -7-4 & 8-3 \end{bmatrix} = \begin{bmatrix} 5 & -1 & -1 \\ -8 & 7 & -8 \\ -1 & -11 & 5 \end{bmatrix}$$

$$B-A = \begin{bmatrix} -1-4 & 0-(-1) & 1-0 \\ 5-(-3) & -2-5 & 2-(-6) \\ 3-2 & 4-(-7) & 3-8 \end{bmatrix} = \begin{bmatrix} -5 & 1 & 1 \\ 8 & -7 & 8 \\ 1 & 11 & -5 \end{bmatrix}$$

Example :

$$A = \begin{bmatrix} 4 & 5 & 0 \\ 0 & -1 & 3 \end{bmatrix} \; B = \begin{bmatrix} 1 & 0 & -1 \\ 4 & -5 & 6 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}$$

Prove that $(A + B) + C = A + (B + C)$

$$A+B = \begin{bmatrix} 4+1 & 5+0 & 0+(-1) \\ 0+4 & -1+(-5) & 3+6 \end{bmatrix} = \begin{bmatrix} 5 & 5 & -1 \\ 4 & -6 & 9 \end{bmatrix}$$

$$B+C = \begin{bmatrix} 1+1 & 0+2 & -1+3 \\ 4+7 & -5+8 & 6+9 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 11 & 3 & 15 \end{bmatrix}$$

$$(A+B)+C = \begin{bmatrix} 5+1 & 5+2 & -1+3 \\ 4+7 & -6+8 & 9+9 \end{bmatrix} = \begin{bmatrix} 6 & 7 & 2 \\ 11 & 2 & 18 \end{bmatrix}$$

$$A+(B+C) = \begin{bmatrix} 4+2 & 5+2 & 0+2 \\ 0+11 & -1+3 & 3+15 \end{bmatrix} = \begin{bmatrix} 6 & 7 & 2 \\ 11 & 2 & 8 \end{bmatrix}$$

$\therefore (A+B)+C = A+(B+C)$

1.4.3 Properties of Matrix Addition

Property 1 : Let A and B be two m x n matrices. Then A + B is also an m x n matrices.

Property 2 : Matrix addition is commutative. Let $A = [a_{ij}]$; $B = [b_{ij}]$ be two m x n matrices. Then $A + B = B + A$.

Property 3 : Matrix addition is associative. Let $A = [a_{ij}]$, $B = [b_{ij}]$ and $C = [c_{ij}]$ are three m x n matrices, then

Space for hints

Space for hints

$$(A+B)+C = A+(B+C)$$

Property 4 : Existence of Identity. Let $O_{m \times n}$ be an $m \times n$ zero matrix and $A=[a_{ij}]$ be also $m \times n$ matrix. Then $A + O_{m \times n} = O_{m \times n} + A = A$ for every matrix A i.e., zero matrix $O_{m \times n}$ is an identity for matrix addition.

Example :

If $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & -2 \\ -1 & 6 \end{bmatrix}$

Show that (i) $(A+B)' = A' + B'$

(ii) $(A-B)' = A' - B'$

$$(A+B) = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} + \begin{bmatrix} 5 & -2 \\ -1 & 6 \end{bmatrix} = \begin{bmatrix} 7 & 1 \\ -2 & 10 \end{bmatrix}$$

$$(A+B)' = \begin{bmatrix} 7 & -2 \\ 1 & 10 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} \qquad A' = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 5 & -2 \\ -1 & 6 \end{bmatrix} \qquad B' = \begin{bmatrix} 5 & -1 \\ -2 & 6 \end{bmatrix}$$

$$A' + B' = \begin{bmatrix} 7 & -2 \\ 1 & 10 \end{bmatrix} = (A+B)'$$

$$A - B = \begin{bmatrix} -3 & 5 \\ 0 & -2 \end{bmatrix}, \quad (A-B)' = \begin{bmatrix} -3 & 0 \\ 5 & -2 \end{bmatrix}$$

$$A' - B' = \begin{bmatrix} -3 & 0 \\ 5 & -2 \end{bmatrix} = (A-B)'$$

1.4.4 Scaler Multiplication of Matrices

$$\text{Let } A = \begin{pmatrix} 5 & 2 \\ -3 & 4 \end{pmatrix}$$

$$A+A+A = \begin{pmatrix} 5 & 2 \\ -3 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 2 \\ -3 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 2 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} 5+5+5 & 2+2+2 \\ -3-3-3 & 4+4+4 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \times 5 & 3 \times 2 \\ 3 \times (-3) & 3 \times 4 \end{pmatrix} = \begin{pmatrix} 15 & 6 \\ -9 & 12 \end{pmatrix}$$

In the above example $A+A+A = 3A$, 3 is a number and A is an arrangement of numbers and we are multiplying two things of different species.

A matrix is said to be multiplied by a scalar if every element in the matrix is multiplied by the same number called scalar. Therefore we can rewrite $A+A+A = 3A$.

Example

Space for hints

(i) $A = \begin{pmatrix} 2 & 0 & 1 \\ 3 & 5 & 4 \end{pmatrix}$ Find 3A.

$$3A = \begin{pmatrix} 3 \times 2 & 3 \times 0 & 3 \times 1 \\ 3 \times 3 & 3 \times 5 & 3 \times 4 \end{pmatrix} = \begin{pmatrix} 6 & 0 & 3 \\ 9 & 15 & 12 \end{pmatrix}$$

(ii) $A = \begin{bmatrix} -1 & -2 & 4 \\ 6 & 7 & 1 \\ 3 & 2 & -4 \end{bmatrix}$ Find 5A

$$5A = \begin{bmatrix} -5 & -10 & 20 \\ 30 & 35 & 5 \\ 15 & 10 & -20 \end{bmatrix}$$

(iii) $B = \begin{bmatrix} 2 & 0 & -5 \\ 3 & 5 & 1 \\ -1 & 0 & -4 \end{bmatrix}$ Find $\frac{2}{3}$ B.

$$\frac{2}{3} B = \begin{bmatrix} \frac{2}{3} \times 2 & \frac{2}{3} \times 0 & \frac{2}{3} \times -5 \\ \frac{2}{3} \times 3 & \frac{2}{3} \times 5 & \frac{2}{3} \times 1 \\ \frac{2}{3} \times -1 & \frac{2}{3} \times 0 & \frac{2}{3} \times -4 \end{bmatrix}$$

$$\begin{bmatrix} \frac{4}{3} & 0 & \frac{10}{3} \\ 2 & \frac{10}{3} & \frac{2}{3} \\ \frac{2}{3} & 0 & \frac{8}{3} \end{bmatrix}$$

iv) $A = \begin{bmatrix} 2 & 6 & -1 \\ 3 & 2 & 5 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$ $C = \begin{bmatrix} 4 & 2 & 3 \\ 0 & 1 & 9 \end{bmatrix}$

Find 5A + 2B – 3C

$$5A = \begin{pmatrix} 10 & 30 & -5 \\ 15 & 10 & 25 \end{pmatrix} \quad 2B = \begin{pmatrix} 2 & 0 & 4 \\ 4 & 2 & -2 \end{pmatrix} \quad 3C = \begin{pmatrix} 12 & 6 & 9 \\ 0 & 3 & 27 \end{pmatrix}$$

$$5A + 2B = \begin{pmatrix} 12 & 30 & -1 \\ 19 & 12 & 23 \end{pmatrix}$$

$$5A + 2B - 3C = \begin{pmatrix} 12 - 12 & 30 - 6 & -1 - 9 \\ 19 - 0 & 12 - 3 & 23 - 27 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 24 & -10 \\ 19 & 9 & -4 \end{pmatrix}$$

Space for hints

1.4.5 Matrix Multiplication

Let us introduce the concept of the product of matrices by means of the following illustrations :

Consider the matrices

$$A = [1 \quad 3 \quad 2 \quad 5]$$

$$B = \begin{bmatrix} 3 \\ -1 \\ 2 \\ 0 \end{bmatrix}$$

A is a matrix with one row and 4 columns. B is a matrix with 4 rows and 1 column. A is called a row matrix and B, a column matrix. A and B can be multiplied by multiplying corresponding elements, i.e., the first element 1 of A is multiplied by the first element 3 of B the second element 3 of A is multiplied by the second element −1 of B and so on ...... and adding the products so obtained.

$$\text{That is, } AB = [1 \quad 3 \quad 2 \quad 5] \begin{bmatrix} 3 \\ -1 \\ 2 \\ 0 \end{bmatrix}$$

$$= [1 \times 3 + 3 \times (-1) + 2 \times 2 + 5 \times 0] = [4]$$

i.e., the product of the row matrix A and the column matrix B is a matrix with a single element.

If there had been one more element in A, it would not have had a corresponding matching element in B. Thus A and B are conformable for multiplication only if the number of columns of A and number of rows of B are equal.

Generalising this to matrices with more than one row and one column, let us consider the following illustration.

$$A = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & -3 \\ 2 & 5 \end{bmatrix}$$

Here the number of columns of A = number of rows of B. Since A is a 2 x 2 matrix and B is also 2 x 2, AB is also 2 x 2, and is defined as follows.

The element in the i-th row and j-th column of AB is obtained by multiplying the i-th row of A and the j-th column of B.

$$\text{Thus, if } AB = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

a_{11} = Product of 1st row of A and the 1st column of B.

Thus $a_{11} = [\,2 \quad 4\,]\begin{bmatrix}-1\\2\end{bmatrix} = 2 \times (-1) + 4 \times 2 = 6$

a_{12} = 1st row of A x second column of B.

$$= [2 \quad 4]\begin{bmatrix}-3\\5\end{bmatrix} = 2 \times (-3) + 4 \times 5$$

$$= -6 + 20 = 14$$

a_{21} = 2nd row of A x 1st column of B.

$$= [0 \quad 3]\begin{bmatrix}-1\\2\end{bmatrix} = 0 \times (-1) + 3 \times 2$$

$$= 0 + 6 = 6$$

a_{22} = 2nd row of A x 2nd column of B.

$$= [0 \quad 3]\begin{bmatrix}-3\\5\end{bmatrix} = 0 \times (-3) + 3 \times 5$$

$$= 0 + 15 = 15$$

Thus $AB = \begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix} = \begin{bmatrix}6 & 14\\ 6 & 15\end{bmatrix}$

Similarly BA can be obtained, since number of columns of B = number of rows of A. Therefore, if

$$BA = \begin{bmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{bmatrix}$$

b_{11} = 1st row of B x 1st column of A

$$= [-1 \quad -3]\begin{bmatrix}2\\0\end{bmatrix} = -1 \times 2 + (-3) \times 0 = -2$$

b_{12} = 1st row of B x 2nd column of A

$$= [-1 \quad -3]\begin{bmatrix}4\\3\end{bmatrix} = -1 \times 4 + (-3) \times 3 = -13$$

b_{21} = 2nd row of B x 1st column of A

$$= [2 \quad 5]\begin{bmatrix}2\\0\end{bmatrix} = 2 \times 2 + 5 \times 0 = 4$$

b_{22} = 2nd row of B x 2nd column of A

$$= [2 \quad 5]\begin{bmatrix}4\\3\end{bmatrix} = 2 \times 4 + 5 \times 3 = 23$$

$$\therefore BA = \begin{bmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{bmatrix} = \begin{bmatrix}-2 & -13\\ 4 & 23\end{bmatrix}$$

Space for hints

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Thus, in this case, even though both, AB and BA exist and are 2 x 2 matrices, we see that

$AB \neq BA.$

Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -4 \\ 1 & 5 \\ -2 & 2 \end{bmatrix}$

Then number of columns of A = number of rows of B. Since A is a 2x3 matrix and B is a 3x2 matrix, A B is a 2x2 matrix. [This is because the first row of A can be multiplied with the 2 columns of B to give two elements in the first row of AB. Similarly the second row of A on multiplication with the two columns of B gives two elements in the second row of AB. Thus AB is a 2x2 matrix.

Let $AB = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

a_{11} = 1st row of A x 1st column of B

$$= [1 \quad 2 \quad 1] \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} = 1 \times 3 + 2 \times 1 + 1 \times (-2) = 3.$$

a_{12} = 1st row of A x 2nd column of B

$$= [1 \quad 2 \quad 1] \begin{bmatrix} -4 \\ 5 \\ 2 \end{bmatrix} = 1 \times (-4) + 2 \times 5 + 1 \times 2 = 8$$

Similarly,

a_{21} = 2nd row of A x 1st column of B

$$= [4 \quad 0 \quad 2] \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} = 8$$

a_{22} = 2nd row of A x 2nd column of B

$$= [4 \quad 0 \quad 2] \begin{bmatrix} -4 \\ 5 \\ 2 \end{bmatrix} = -12$$

Thus $AB = \begin{bmatrix} 3 & 8 \\ 8 & -12 \end{bmatrix}$

Now B has 2 columns and A has 2 rows. ∴ BA is also defined. Also, since B is a 3x2 matrix and A is a 2x3 matrix the product BA is a 3x3 matrix.

∴ If $BA = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$, then

b_{11} = 1st row of B x 1st column of A

$$= 3 \times 1 + (-4) \times 4 = -13$$

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b_{12} = 1st row of B x 2nd column of A

$= 3 \times 2 + (-4) \times 0 = 6$

b_{13} = 1st row of B x 3rd column of A

$= 3 \times 1 + (-4) \times 2 = -5$

b_{21} = 2nd row of B x 1st column of A

$= 1 \times 1 + 5 \times 4 = 21$

b_{22} = 2nd row of B x 2nd column of A

$= 1 \times 2 + 5 \times 0 = 2$

b_{23} = 2nd row of B x 3rd column of A

$= 1 \times 1 + 5 \times 2 = 11$

b_{31} = 3rd row of B x 1st column of A

$= -2 \times 1 + 2 \times 4 = 6$

b_{32} = 3rd row of B x 2nd column of A

$= -2 \times 2 + 2 \times 0 = -4$

b_{33} = 3rd row of B x 3rd column of A

$= -2 \times 1 + 2 \times 2 = 2$

$$\text{Thus } BA = \begin{bmatrix} -13 & 6 & -5 \\ 21 & 2 & 11 \\ 6 & -4 & 2 \end{bmatrix}$$

∴ Even though both AB and BA exist, they are not of the same order. Thus if an m x n matrix A is multiplied by a n x m matrix B, the product AB is of order m x m and the product BA is of order n x n.

$$\text{Let } A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix}$$

Since A is 2 x 2 and B is also 2 x 2, both AB and BA exist.

$$AB = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \times 1 + 0 \times 3 & 1 \times 0 + \quad \times 4 \\ 2 \times 1 + 3 \times 3 & 2 \times 0 + 3 \times 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 11 & 12 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$$

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$$= \begin{bmatrix} 1 \times 1 + 0 \times 2 & 1 \times 0 + 0 \times 3 \\ 3 \times 1 + 4 \times 2 & 3 \times 0 + 4 \times 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 11 & 12 \end{bmatrix}$$

Thus AB = BA in this case.

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \times 1 + 1 \times (-1) & 1 \times (-1) + 1 \times 1 \\ 1 \times 1 + 1 \times (-1) & 1 \times (-1) + 1 \times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ AB is the zero matrix even though both A and B are non zero matrices.

Let $A = \begin{bmatrix} 3 & 0 \\ 1 & 1 \\ 5 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 7 \\ 6 & 8 \end{bmatrix}$

$$AB = \begin{bmatrix} 3 & 0 \\ 1 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 4 & 7 \\ 6 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 3\times4 + 0\times6 + 3\times7 + 0\times8 \\ 1\times4 + 1\times6 + 1\times7 + 1\times8 \\ 5\times4 + 2\times6 + 5\times7 + 2\times8 \end{bmatrix} = \begin{bmatrix} 12 & 21 \\ 10 & 15 \\ 32 & 51 \end{bmatrix}$$

The product BA is however not defined in this case, as number of columns of B = 2, whereas number of rows of A = 3.

Based on the examples given so far, multiplication of two matrices A and B in general is defined as follows :

$$\text{Let } A = [a_{ij}]_{m \times n} \quad \text{and} \quad B = [b_{ij}]_{n \times p}$$

Number of columns in A and number of rows in B are equal. Now, if the product of the i^{th} row in A and the j^{th} column in B is denoted by c_{ij}, then $C = [c_{ij}]$ is called the product of the two matrices A and B. The order of the product matrix C will be m x p and

$$c_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \ldots\ldots + a_{in} b_{nj}$$

This can also be written as

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

In the multiplication of two matrices, the following points are to be noted:

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1.4.6 Points to be noted in Matrix Multiplication

(i) The product matrix AB can be obtained only when the number of columns in A is equal to the number of rows in B. In such a case, A is said to be conformable to B for multiplication.

(ii) Number of rows in AB will be equal to the number of rows in A and the number of columns in AB will be equal to the number of columns in B.

(iii) When A is conformable to B for multiplication it is not necessary that B is conformable to A for multiplication.

(iv) Only when the number of columns in A is equal to the number of rows in B and the number of rows in A is equal to the number of columns in B, A will be conformable to B as well as B will be conformable to A for multiplication. That is, if A is a matrix of order mxn and B of order n x m, then both AB and BA exist.

(v) When both AB and BA exist,

(i) AB need not be equal to BA and

(ii) the order of AB need not be equal to the order of BA.

(vi) In the product AB, A is called 'premultiplier and B is called 'post multiplier'.

(vii) When no one of the two matrices, A, B is a zero matrix, we may get their product matrix, AB as a zero matrix.

Example :

If $A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ find AB.

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

$$\therefore AB = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \times 1 + 3 \times 2 & 2 \times 2 + 3 \times (-1) \\ 3 \times 1 + 4 \times 2 & 3 \times 2 + 4 \times (-1) \end{bmatrix}$$

$$= \begin{bmatrix} 2 + 6 & 4 - 3 \\ 3 + 8 & 6 - 4 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 1 \\ 11 & 2 \end{bmatrix}$$

Example :

Find the product of the following matrices.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix}$$

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$$AB = \begin{bmatrix} 1 \times 10 + 2 \times 0 & 1 \times 0 + 2 \times 1 \\ 3 \times 10 + 4 \times 0 & 3 \times 0 + 4 \times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 10 + 0 & 0 + 2 \\ 30 + 0 & 0 + 4 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 2 \\ 30 & 4 \end{bmatrix}$$

Example :

If $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ find AB.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\therefore AB = \begin{bmatrix} 1 \times 1 + 0 \times 4 + 0 \times 7 & 1 \times 2 + 0 \times 5 + 0 \times 8 & 1 \times 3 + 0 \times 6 + 0 \times 9 \\ (-1) \times 1 + 1 \times 4 + 0 \times 7 & (-1) \times 2 + 1 \times 5 + 0 \times 8 & (-1) \times 3 + 1 \times 6 + 0 \times 9 \\ 0 \times 1 + 0 \times 4 + 1 \times 7 & 0 \times 2 + 0 \times 5 + 1 \times 8 & 0 \times 3 + 0 \times 6 + 1 \times 9 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0 & 2+0+0 & 3+0+0 \\ -1+4+0 & -2+5+0 & -3+6+0 \\ 0+0+7 & 0+0+8 & 0+0+9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 7 & 8 & 9 \end{bmatrix}$$

Example :

If $A = \begin{bmatrix} 1 & 3 & -5 \\ 2 & 4 & 0 \end{bmatrix}$ and

$B = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 3 & 1 \\ 0 & 1 & 6 \end{bmatrix}$ prove that $(AB)^T = B^T A^T$.

$$A = \begin{bmatrix} 1 & 3 & -5 \\ 2 & 4 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 3 & 1 \\ 0 & 1 & 6 \end{bmatrix}$$

$$\therefore AB = \begin{bmatrix} 1 \times 1 + 3(-4) + (-5) \times 0 & 1 \times 2 + 3 \times 3 + (-5) \times 1 & 1 \times 3 + 3 \times 1 + (-5) \times 6 \\ 2 \times 1 + 4 \times (-4) + 0 \times 0 & 2 \times 2 + 4 \times 3 + 0 \times 1 & 2 \times 3 + 4 \times 1 + 0 \times 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1-12+0 & 2+9-5 & 3+3-30 \\ 2-16+0 & 4+12+0 & 6+4+0 \end{bmatrix}$$

$$= \begin{bmatrix} -11 & 6 & -24 \\ -14 & 16 & 10 \end{bmatrix}$$

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$$(AB)^T = \begin{bmatrix} -11 & -14 \\ 6 & 16 \\ -24 & 10 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 1 & -4 & 0 \\ 2 & 3 & 1 \\ 3 & 1 & 6 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ -5 & 0 \end{bmatrix}$$

$$\therefore B^T A^T = \begin{bmatrix} 1 \times 1 + (-4) \times 3 + 0 \times (-5) & 1 \times 2 + (-4) \times 4 + 0 \times 0 \\ 2 \times 1 + 3 \times 3 + 1 \times (-5) & 2 \times 2 + 3 \times 4 + 1 \times 0 \\ 3 \times 1 + 1 \times 3 + 6 \times (-5) & 3 \times 2 + 1 \times 4 + 6 \times 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - 12 + 0 & 2 - 16 + 0 \\ 2 + 9 - 5 & 4 + 12 + 0 \\ 3 + 3 - 30 & 6 + 4 + 0 \end{bmatrix}$$

$$= \begin{bmatrix} -11 & -14 \\ 6 & 16 \\ -24 & 10 \end{bmatrix} = (AB)^T$$

$\therefore (AB)^T = B^T A^T$

2. Determinants

2.1 Meaning and Notation

Consider the linear equations in two variables, x and y

$ax + by = 0$ .... (1)

$cx + dy = 0$ .... (2)

These two equations can be rewritten as follows : $x = \frac{-by}{a}$ and $x = \frac{-dy}{c}$

$\therefore \frac{-by}{a} = \frac{-dy}{c}$. That is, $\frac{-b}{a} = \frac{-d}{c}$. That is, $-bc = -ad$.

That is, $ad - bc = 0$. The expression $(ad - bc)$ is called a determinant of order 2 and is written as $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

2.2 Matrix and Determinant - Relationship between them

Consider the square matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ of order 2. In this matrix instead of the brackets, if we put two vertical lines, it is called a determinant associated with the given matrix A. That is, $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is called the determinant of the square matrix A and

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it is denoted by | A | or by det. A This is assigned the specific value (ad – bc) which is derived according to the specific rule given in subsequent paragraphs, under the topic "Evaluation of determinants". Thus with every square matrix of order 2 is associated a numerical value called the determinant value of the matrix.

Similarly, let us consider a matrix of order n as follows.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & \cdots & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & \cdots & \cdots & a_{nn} \end{bmatrix}$$

The determinant associated with this matrix is written as follows.

$$|A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & \cdots & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & \cdots & \cdots & a_{nn} \end{vmatrix}$$

In the evaluation of determinants of higher order certain concepts are used. These are explained below.

2.3 Minor determinant

Consider the n th order determinant.

$$|A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ a_{n1} & a_{n2} & a_{nj} & \cdots & \cdots & \cdots & a_{nn} \end{vmatrix}$$

In this determinant, if we remove the i^{th} row and j^{th} column, we will get a determinant of order (n–1). This determinant of order (n–1) is called the 'minor of a_{ij}' and it is denoted by M_{ij}.

$$\text{Let } = A \begin{bmatrix} a & b & c \\ x & y & z \\ u & v & w \end{bmatrix}$$

In this matrix, minor of a is obtained by removing the row and column passing through a. That is, removing the first row and first column the minor of a is obtained.

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$$\text{Minor of } a = M_{11} = \begin{vmatrix} y & z \\ v & w \end{vmatrix}$$

$$\text{Similarly, minor of } y = M_{22} = \begin{vmatrix} a & c \\ u & w \end{vmatrix}$$

$$\text{Minor of } w = M_{33} = \begin{vmatrix} a & b \\ x & y \end{vmatrix}$$

$$\text{Minor of } z = M_{23} = \begin{vmatrix} a & b \\ u & v \end{vmatrix}$$

Example :

$$\text{Let } A = \begin{bmatrix} 1 & -1 & 0 & 3 \\ 4 & 6 & 5 & -7 \\ 2 & -10 & -3 & 6 \\ -7 & 11 & 8 & -4 \end{bmatrix}$$

Write down the minors of the elements in the places (1, 1) (2, 4) and (4,1).

(1,1) element is 1. Its minor is

$$M_{11} = \begin{vmatrix} 6 & 5 & -7 \\ -10 & -3 & 6 \\ 11 & 8 & -4 \end{vmatrix}$$

(2, 4) element is –7 and its minor is

$$M_{24} = \begin{vmatrix} 1 & -1 & 0 \\ 2 & -10 & -3 \\ -7 & 11 & 8 \end{vmatrix}$$

(4,1) element is –7 and its minor is

$$M_{41} = \begin{vmatrix} -1 & 0 & 3 \\ 6 & 5 & -7 \\ -10 & -3 & 6 \end{vmatrix}$$

2.4 Cofactor

The minor determinant M_{ij} when multiplied by $(-1)^{i+j}$ is called the cofactor of a_{ij} and it is denoted by A_{ij}.

$$\therefore \; A_{ij} = (-1)^{i+j} M_{ij}$$

For the example already given let us find out the cofactors

$$A = \begin{bmatrix} a & b & c \\ x & y & z \\ u & v & w \end{bmatrix}$$

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$$\text{Cofactor of } a = A_{11} = (-1)^{1+1} M_{11}$$
$$= (-1)^2 \begin{vmatrix} y & z \\ v & w \end{vmatrix}$$
$$= + \begin{vmatrix} y & z \\ v & w \end{vmatrix}$$

$$\text{Cofactor of } b = A_{12} = (-1)^{1+2} M_{12}$$
$$= (-1)^3 M_{12}$$
$$= - \begin{vmatrix} x & z \\ u & w \end{vmatrix}$$

$$\text{Cofactor of } c = A_{13} = (-1)^{1+3} M_{13}$$
$$= (-1)^4 M_{13}$$
$$= + \begin{vmatrix} x & y \\ u & v \end{vmatrix}$$

$$\text{Cofactor of } x = A_{21} = (-1)^{2+1} M_{21}$$
$$= (-1)^3 M_{21}$$
$$= - \begin{vmatrix} b & c \\ v & w \end{vmatrix}$$

$$\text{Cofactor of } y = A_{22} = (-1)^{2+2} M_{22}$$
$$= (-1)^4 M_{22}$$
$$= + \begin{vmatrix} a & c \\ u & w \end{vmatrix}$$

$$\text{Cofactor of } z = A_{23} = (-1)^{2+3} M_{23}$$
$$= (-1)^5 M_{23}$$
$$= - \begin{vmatrix} a & b \\ u & v \end{vmatrix}$$

$$\text{Cofactor of } u = A_{31} = (-1)^{3+1} M_{31}$$
$$= (-1)^4 M_{31}$$
$$= + \begin{vmatrix} b & c \\ y & z \end{vmatrix}$$

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Cofactor of $v = A_{32} = (-1)^{3+2}\ M_{32}$

$$= (-1)^5\ M_{32}$$

$$= -\begin{vmatrix} a & c \\ x & z \end{vmatrix}$$

Cofactor of $w = A_{33} = (-1)^{3+3}\ M_{33}$

$$= (-1)^6\ M_{33}$$

$$= +\begin{vmatrix} a & b \\ x & y \end{vmatrix}$$

Example :

If $A = \begin{vmatrix} 4 & -2 & -3 & 3 \\ -1 & 0 & 5 & 11 \\ 6 & 1 & -4 & 8 \\ 1 & -1 & 2 & 9 \end{vmatrix}$ find A_{11}, A_{12}, A_{21} and A_{42}.

$$A_{11} = (-1)^{1+1}\ M_{11} = (-1)^2\ M_{11} = M_{11}$$

$$= \begin{vmatrix} 0 & 5 & 11 \\ 1 & -4 & 8 \\ -1 & 2 & 9 \end{vmatrix}$$

$$A_{12} = (-1)^{1+2}\ M_{12} = (-1)^3\ M_{12} = -M_{12}$$

$$= -\begin{vmatrix} -1 & 5 & 11 \\ 6 & -4 & 8 \\ 1 & 2 & 9 \end{vmatrix} = \begin{vmatrix} 1 & -5 & -11 \\ -6 & 4 & -8 \\ -1 & -2 & -9 \end{vmatrix}$$

$$A_{21} = (-1)^{2+1}\ M_{21} = (-1)^3\ M_{21} = -M_{21}$$

$$= -\begin{vmatrix} -2 & -3 & 3 \\ 1 & -4 & 8 \\ -1 & 2 & 9 \end{vmatrix} = \begin{vmatrix} 2 & 3 & -3 \\ 1 & 4 & -8 \\ 1 & -2 & -9 \end{vmatrix}$$

$$A_{42} = (-1)^{4+2}\ M_{42} = (-1)^6\ M_{42} = M_{42}$$

$$= \begin{vmatrix} 4 & -3 & 3 \\ -1 & 5 & 11 \\ 6 & -4 & 8 \end{vmatrix}$$

2.5 Evaluation of determinants

2.5.1 Evaluation of second order determinant :

The value of the second order determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is got by subtracting the

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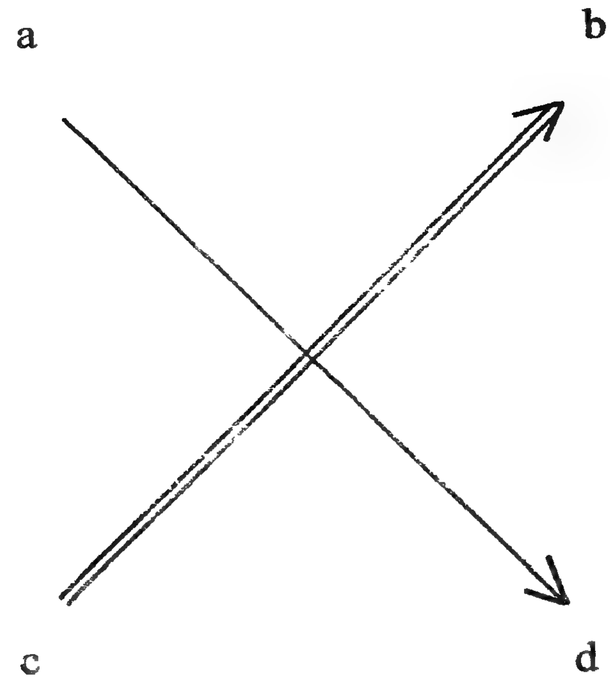
product b.c of the elements along the double arrow from the product a.d of the elements along the single arrow indicated below :



That is, (ad – bc) is the value of the given determinant of order 2.

Example

The value of $\begin{vmatrix} 2 & 3 \\ -2 & 1 \end{vmatrix}$ is $2 \times 1 - 3\,(-2) = 2 + 6 = 8$

The value of $\begin{vmatrix} 0 & -3 \\ -2 & 1 \end{vmatrix}$ is $0 \times 1 - (-3)\,(-2) = 0 - 6 = -6$

The value of $\begin{vmatrix} 0 & 0 \\ 3 & 2 \end{vmatrix}$ is $0 \times 2 - 3 \times 0 = 0 - 0 = 0$

The value of $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$ is $1 \times 1 - 0 \times 0 = 1 - 0 = 1$

2.5.2 Evaluation of a determinant of order 3 :

When a third order determinant is given, any row (or any column) of the determinant is chosen first. Secondly, each element in the row (or column) chosen is multiplied by its cofactor and the products obtained are summed up. This sum gives us the required value of the determinant Let us illustrate this with an example.

$$\text{Let } A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Let us choose the first row. The elements in the first row are a_{11}, a_{12} and a_{13}. The cofactors of these elements are A_{11}, A_{12} and A_{13} respectively. Now the value of the given determinant is $a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13}$.

The value of a determinant is generally denoted by D (read as delta).

$$\therefore \ \Delta = a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13}.$$

Instead of the first row, if the first column is chosen.

$$\Delta = a_{11} A_{11} + a_{21} A_{21} + a_{31} A_{31}.$$

Similarly we can find out the value of determinant by choosing any other row or column also.

Example:

Find the value of the determinant $\begin{vmatrix} 0 & -2 & 1 \\ 6 & 3 & 0 \\ 4 & -1 & 1 \end{vmatrix}$

Let us choose the first row and find out the value of the determinant.

$$a_{11} = 0 \qquad a_{12} = -2 \qquad a_{13} = 1$$

$$A_{11} = (-1)^{1+1} M_{11} = (-1)^2 M_{11} = +\begin{vmatrix} 3 & 0 \\ -1 & 1 \end{vmatrix}$$

$$= 3 \times 1 - (-1) \times 0 = 3 + 0 = 3$$

$$A_{12} = (-1)^{1+2} M_{12} = (-1)^3 M_{12} = -M_{12}$$

$$= -\begin{vmatrix} 6 & 0 \\ 4 & 1 \end{vmatrix} = -(6 \times 1 - 4 \times 0) = -6 + 0 = -6$$

$$A_{13} = (-1)^{1+3} M_{13} = (-1)^4 M_{13} = +M_{13}$$

$$= \begin{vmatrix} 6 & 3 \\ 4 & -1 \end{vmatrix} = 6 \times (-1) - 4 \times 3 = -6 - 12 = -18$$

$$\Delta = a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13}$$

$$= 0 \times 3 + (-2)(-6) + 1(-18)$$

$$= 0 + 12 - 18 = -6.$$

All the calculations given so far are in an elaborate manner. The same calculations may be presented in a shortened form as follows

$$\Delta = a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13}$$

$$= 0 \times (-1)^{1+1}\begin{vmatrix} 3 & 0 \\ -1 & 1 \end{vmatrix} + (-2)(-1)^{1+2}\begin{vmatrix} 6 & 0 \\ 4 & 1 \end{vmatrix} + (1)(-1)^{1+3}\begin{vmatrix} 6 & 3 \\ 4 & -1 \end{vmatrix}$$

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$$= 0 + 2\,(6 \times 1 - 4 \times 0) + [6 \times (-1) - 4 \times 3]$$

$$= 2 \times 6 - 6 - 12$$

$$= 12 - 18 = -6.$$

Instead of the first row, if we choose the first column,

$$\Delta = a_{11} A_{11} + a_{21} A_{21} + a_{31} A_{31}$$

$$= 0 \times (-1)^{1+1} \begin{vmatrix} 3 & 0 \\ -1 & 1 \end{vmatrix} + 6\,(-1)^{2+1} \begin{vmatrix} -2 & 1 \\ -1 & 1 \end{vmatrix} + 4\,(-1)^{3+1} \begin{vmatrix} -2 & 1 \\ 3 & 0 \end{vmatrix}$$

$$= 0 - 6\,[(-2) \times 1 - (-1) \times 1] + 4\,[(-2) \times 0 - 3 \times 1]$$

$$= -6\,(-2 + 1) + 4(0 - 3)$$

$$= -6\,(-1) + 4\,(-3) = 6 - 12 = -6.$$

Similarly, any other row or column may also be chosen and the value of the determinant may be calculated. In all the cases the answer will be the same.

2.5.3 Evaluation of a determinant of higher order :

The method used to evaluate the determinant of order 3 can be extended to evaluate the determinant of any order. Generally, the value of the determinant of order n is written as follows :

$$\Delta = a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13} + \ldots\ldots\ldots + a_{1n} A_{1n}$$

Example :

Evaluate $\begin{vmatrix} 4 & 5 & 0 & -3 \\ 3 & -4 & 1 & 6 \\ 2 & 7 & 8 & 0 \\ 3 & 5 & 0 & 0 \end{vmatrix}$

Let us choose the first row and find out the value of the determinant.

$$\therefore\ D = 4 \times (-1)^{1+1} \begin{vmatrix} -4 & 1 & 6 \\ 7 & 8 & 0 \\ 5 & 0 & 0 \end{vmatrix} + 5\,(-1)^{1+2} \begin{vmatrix} 3 & 1 & 6 \\ 2 & 8 & 0 \\ 3 & 0 & 0 \end{vmatrix} + 0\,(-1)^{1+3} \begin{vmatrix} 3 & -4 & 6 \\ 2 & 7 & 0 \\ 3 & 5 & 0 \end{vmatrix}$$

$$+ (-3)\,(-1)^{1+4} \begin{vmatrix} 3 & -4 & 1 \\ 2 & 7 & 8 \\ 3 & 5 & 0 \end{vmatrix}$$

In the above expansion each term has a determinant of order 3. These determinants can again be expanded using any row or columns.

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$$\Delta = 4\begin{vmatrix} -4 & 1 & 6 \\ 7 & 8 & 0 \\ 5 & 0 & 0 \end{vmatrix} - 5\begin{vmatrix} 3 & 1 & 6 \\ 2 & 8 & 0 \\ 3 & 0 & 0 \end{vmatrix} + 0 + 3\begin{vmatrix} 3 & -4 & 1 \\ 2 & 7 & 8 \\ 3 & 5 & 0 \end{vmatrix}$$

$$= 4\left\{-4(-1)^{1+1}\begin{vmatrix} 8 & 0 \\ 0 & 0 \end{vmatrix} + 1(-1)^{1+2}\begin{vmatrix} 7 & 0 \\ 5 & 0 \end{vmatrix} + 6(-1)^{1+3}\begin{vmatrix} 7 & 8 \\ 5 & 0 \end{vmatrix}\right\}$$

$$-5\left\{3(-1)^{1+1}\begin{vmatrix} 8 & 0 \\ 0 & 0 \end{vmatrix} + 1(-1)^{1+2}\begin{vmatrix} 2 & 0 \\ 3 & 0 \end{vmatrix} + 6(-1)^{1+3}\begin{vmatrix} 2 & 8 \\ 3 & 0 \end{vmatrix}\right\}$$

$$+3\left\{3(-1)^{1+1}\begin{vmatrix} 7 & 8 \\ 5 & 0 \end{vmatrix} - 4(-1)^{1+2}\begin{vmatrix} 2 & 8 \\ 3 & 0 \end{vmatrix} + 1(-1)^{1+3}\begin{vmatrix} 2 & 7 \\ 3 & 5 \end{vmatrix}\right\}$$

$$= 4\{-4 \times 0 - 1 \times 0 + 6(0-40)\} - 5\{3 \times 0 - 1 \times 0 + 6(0-24)\}$$
$$+ 3\{3(0-40) + 4(0-24) + (10-21)\}$$

$$= 4(-240) - 5(-144) + 3(-120 - 96 - 11)$$

$$= -960 + 720 + 3(-227)$$

$$= -240 - 681$$

$$= -921.$$

In the given determinant, both in the last column and in the last row, two elements are equal to zero. Therefore, if we try to find out the value of the determinant by expanding the determinant using either the last column or the last row, our computations will be rather easy. This we have illustrated below. We have expanded the determinant using the last row. (The students may try the expansion through the last column and verify whether the same answer is obtained).

$$\Delta = 3 \times (-1)^{4+1}\begin{vmatrix} 5 & 0 & -3 \\ -4 & 1 & 6 \\ 7 & 8 & 0 \end{vmatrix} + 5 \times (-1)^{4+2}\begin{vmatrix} 4 & 0 & -3 \\ 3 & 1 & 6 \\ 2 & 8 & 0 \end{vmatrix}$$

$$= -3\begin{vmatrix} 5 & 0 & -3 \\ -4 & 1 & 6 \\ 7 & 8 & 0 \end{vmatrix} + 5\begin{vmatrix} 4 & 0 & -3 \\ 3 & 1 & 6 \\ 2 & 8 & 0 \end{vmatrix}$$

Now, in each of the above 3rd order determinants, one element in the first row is zero. So, we can use the first row to expand the determinant. (In both the determinants, the last row and the last column also contain zero element. Either of these two may also be selected by the students to expand the determinants and the answers may be verified).

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$$\therefore\ \Delta = -3\left\{5x(-1)^{1+1}\begin{vmatrix}1 & 6\\ 8 & 0\end{vmatrix} - 3x(-1)^{1+3}\begin{vmatrix}-4 & 1\\ 7 & 8\end{vmatrix}\right\}$$

$$+ 5\left\{4x(-1)^{1+1}\begin{vmatrix}1 & 6\\ 8 & 0\end{vmatrix} - 3\,(-1)^{1+3}\begin{vmatrix}3 & 1\\ 2 & 8\end{vmatrix}\right\}$$

$$= -3\ \{5(0-48) - 3(-32-7)\}\ +\ 5\ \{4(0-48) - 3(24-2)\}$$

$$= -3\ \{-240 + 117\}\ +\ 5\ \{-192 - 66\}$$

$$= -3 \text{ x } (-123)\ +\ 5 \text{ x } (-258)$$

$$= 369 - 1290\ =\ -921.$$

It is to be noted by the students that we have got the same answer earlier also.

2.5.4 Evaluation of a determinant with respect to triangular matrix :

Determinants associated with triangular matrices will nave their values equal to the respective products of the elements along their principal diagonals. That is,

$$D\ =\ a_{11},\ a_{22},\ a_{33}\ \ldots\ldots\ a_{nn}$$

Example :

$$\text{Evaluate}\quad\begin{vmatrix}1 & 0 & 0\\ 3 & 4 & 0\\ -2 & 0 & -3\end{vmatrix}$$

$$\Delta\ =\ 1 \text{ x } 4 \text{ x } (-3)\ =\ -12$$

(**Note:** You can verify this answer by the usual procedure).

Example :

$$\text{Evaluate}\quad\begin{vmatrix}3 & 1 & -2 & -4\\ 0 & 4 & -1 & 6\\ 0 & 0 & 5 & 2\\ 0 & 0 & 0 & 1\end{vmatrix}$$

$$\Delta\ =\ 3 \text{ x } 4 \text{ x } 5 \text{ x } 1\ =\ 60.$$

The determinant value of a unit matrix of any order is always equal to 1.
For instance,

$$\begin{vmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{vmatrix} = 1$$

If a determinant has two or more identical rows (or columns) its value will be equal to zero.

For example, consider the determinant.

$$\begin{vmatrix} 5 & 1 & 6 \\ 5 & 1 & 6 \\ 15 & 8 & -2 \end{vmatrix}$$

$$D = 5(-1)^{1+1}\begin{vmatrix} 1 & 6 \\ 8 & -2 \end{vmatrix} + 1(-1)^{1+2}\begin{vmatrix} 5 & 6 \\ 15 & -2 \end{vmatrix} + 6(-1)^{1+3}\begin{vmatrix} 5 & 1 \\ 15 & 8 \end{vmatrix}$$

$$= 5\,\{1(-2) - 8 \times 6\} - \{5(-2) - 15 \times 6\} + 6\,\{5 \times 8 - 15 \times 1\}$$

$$= 5\,\{-2 - 48\} - \{-10 - 90\} + 6\{40 - 15\}$$

$$= 5\,(-50) - (-100) + 6\,(25)$$

$$= -250 + 100 + 150 = -250 + 250 = 0.$$

3. Solving Simultaneous Linear Equations

Simultaneous linear equations may be solved by two methods namely,

(i) Matrix method and

(ii) Determinants method

3.1 Matrix Method :

In this method, the given set of simultaneous linear equations are represented by means of matrices as detailed below :

Suppose we are given n equations in n variables as follows

$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n = b_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{n1}x_1 + a_{n2}x_2 + \ldots\ldots + a_{nn}x_n = b_n$$

The expressions on the LHS of the above equations may be represented as a product of two matrices as AX where

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots\ldots & a_{nn} \end{bmatrix}_{n \times n} \quad \text{and } X = \begin{bmatrix} x_1 \\ x_2 \\ : \\ : \\ x_n \end{bmatrix}_{n \times 1}$$

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The RHS of the n equations may be represented by a matrix, $B = \begin{bmatrix} b_1 \\ b_2 \\ : \\ : \\ b_n \end{bmatrix}_{n \times 1}$

Therefore, the given set of n equations may be written in a compact form as

$$AX = B.$$

Here, matrix A is called **'coefficient-Matrix'** ; x is called **'unknowns' Matrix'** ; and B is called **'constants' Matrix'**.

Solving the given equations means finding the values of the elements of the unknowns' Matrix namely, X. The unknowns' Matrix is got by using a concept known as **'Matrix Inversion'** as follows:

$$X = A^{-1} B$$

Here A^{-1} is called the **'Inverse Matrix of A'**.

The inverse matrix, A^{-1} is defined as

$$A^{-1} = \frac{1}{|A|} \cdot \text{Adj. } A$$

where $|A|$ = Determinant of A

Adj. A = Adjoint matrix of A

= (Cofactor matrix of A)^T

= $(c_{ij})^T$

3.1.1 Finding Inverse Matrix

The method of finding out inverse matric is illustrated below with the help of a (2 x 2) matrix.

$$\text{Let } A = \begin{bmatrix} 2 & 2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 2 \\ 3 & 5 \end{vmatrix} = 10 - 6 = 4.$$

Check your Progress

10. What is meant by co-efficient Matrix?

Since A is a (2 x 2) matrix cofactor matrix (c_{ij}) will also be a (2 x 2) matrix and its elements are as follows.

$$c_{11} = \text{Cofactor of } a_{11} = (-1)^{1+1} M_{11} = 5$$

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c_{12} = Cofactor of $a_{12} = (-1)^{1+2}\ M_{12} = -3$

c_{21} = Cofactor of $a_{21} = (-1)^{2+1}\ M_{21} = -2$

c_{22} = Cofactor of $a_{22} = (-1)^{2+2}\ M_{22} = 2.$

where M stands for Minor.

$$\therefore\ (c_{ij}) = \begin{bmatrix} 5 & -3 \\ -2 & 2 \end{bmatrix}$$

$$\text{Adj. A} = (c_{ij})^T = \begin{bmatrix} 5 & -3 \\ -2 & 2 \end{bmatrix}^T = \begin{bmatrix} 5 & -2 \\ -3 & 2 \end{bmatrix}$$

$$\therefore\ A^{-1} = \frac{1}{|A|} \cdot \text{Adj. A}$$

$$= \frac{1}{4}\begin{bmatrix} 5 & -2 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} \frac{5}{4} & \frac{-2}{4} \\ \frac{-3}{4} & \frac{2}{4} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{4} & \frac{-1}{2} \\ \frac{-3}{4} & \frac{1}{2} \end{bmatrix}$$

Example :

Find the inverse of the matrix.

$$A = \begin{bmatrix} 4 & 0 & 2 \\ 2 & 10 & 2 \\ 3 & 9 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \cdot \text{Adj. A}$$

$$|A| = \begin{bmatrix} 4 & 0 & 2 \\ 2 & 10 & 2 \\ 3 & 9 & 1 \end{bmatrix}$$

Evaluating $|A|$ by the I row, we get

$$|A| = 4\begin{vmatrix} 10 & 2 \\ 9 & 1 \end{vmatrix} + 2\begin{vmatrix} 2 & 10 \\ 3 & 9 \end{vmatrix}$$

$$= 4\,(10 - 18) + 2\,(18 - 30)$$

$$= 4\,(-8) + 2\,(-12) = -32 - 24 = -56$$

$$\text{Adj. A} = (c_{ij})^T$$

As A is (3 x 3) matrix, (c_{ij}) is also a (3 x 3) matrix.

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Now,

$$c_{11} = (-1)^{1+1} \begin{vmatrix} 10 & 2 \\ 9 & 1 \end{vmatrix} = 10-18 = -8$$

$$c_{12} = (-1)^{1+2} \begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix} = -(2-6) = 4$$

$$c_{13} = (-1)^{1+3} \begin{vmatrix} 2 & 10 \\ 3 & 9 \end{vmatrix} = (18-30) = -12$$

$$c_{21} = (-1)^{2+1} \begin{vmatrix} 0 & 2 \\ 9 & 1 \end{vmatrix} = -(0-18) = 18$$

$$c_{22} = (-1)^{2+2} \begin{vmatrix} 4 & 2 \\ 3 & 1 \end{vmatrix} = (4-6) = -2$$

$$c_{23} = (-1)^{2+3} \begin{vmatrix} 4 & 0 \\ 3 & 9 \end{vmatrix} = -(36-0) = -36$$

$$c_{31} = (-1)^{3+1} \begin{vmatrix} 0 & 2 \\ 10 & 2 \end{vmatrix} = (0-20) = -20$$

$$c_{32} = (-1)^{3+2} \begin{vmatrix} 4 & 2 \\ 2 & 2 \end{vmatrix} = -(8-4) = -4$$

$$c_{33} = (-1)^{3+3} \begin{vmatrix} 4 & 0 \\ 2 & 10 \end{vmatrix} = (40-0) = 40.$$

$$\therefore \ (c_{ij}) = \begin{bmatrix} -8 & 4 & -12 \\ 18 & -2 & -36 \\ -20 & -4 & 40 \end{bmatrix}$$

$$\text{Adj. } A = (c_{ij})^T = \begin{bmatrix} -8 & 18 & -20 \\ 4 & -2 & -4 \\ -12 & -36 & 40 \end{bmatrix}$$

$$\therefore \ A^{-1} = \frac{1}{-56} \begin{bmatrix} -8 & 18 & -20 \\ 4 & -2 & -4 \\ -12 & -36 & 40 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-8}{-56} & \frac{18}{-56} & \frac{-20}{-56} \\ \frac{4}{-56} & \frac{-2}{-56} & \frac{-4}{-56} \\ \frac{-12}{-56} & \frac{-36}{-56} & \frac{40}{-56} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{7} & \frac{-9}{28} & \frac{5}{14} \\ \frac{-1}{14} & \frac{1}{28} & \frac{1}{14} \\ \frac{3}{14} & \frac{18}{28} & \frac{-5}{7} \end{bmatrix}$$

3.1.2 Solving Equation by Matrix Method - Illustration :

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Let the given equations be

$2x - y = 5$

$x + y = 7$

First we rewrite the above equations with the help of matrix notation as follows:

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

Put $A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$

$\therefore\ AX = B$

$$X = A^{-1} B$$

$$A^{-1} = \frac{1}{|A|}. \text{ Adj. } A$$

$$|A| = \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = (2 + 1) = 3$$

$$\text{Adj. } A = (c_{ij})^T = \begin{vmatrix} 1 & -1 \\ -(-1) & 2 \end{vmatrix}^T = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix}^T = \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix}$$

$$\therefore\ A^{-1} = \frac{1}{3}\begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{-1}{3} & \frac{2}{3} \end{vmatrix}$$

$$X = A^{-1} B = \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{-1}{3} & \frac{2}{3} \end{vmatrix}\begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} \times 5 + \frac{1}{3} \times 7 \\ \frac{-1}{3} \times 5 + \frac{2}{3} \times 7 \end{bmatrix} = \begin{bmatrix} \frac{12}{3} \\ \frac{9}{3} \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \qquad \therefore\ x = 4 \text{ and } y = 3$$

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Example :

Solve the following equations by the matrix inversion method.

$$x + 3y + 4z = 7$$
$$4x + 2y + 3z = 10$$
$$x + y + z = 3$$

Here $A = \begin{bmatrix} 1 & 3 & 4 \\ 4 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}_{3x3}$; $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3x1}$ and $B = \begin{bmatrix} 7 \\ 10 \\ 3 \end{bmatrix}_{3x1}$

$$X = A^{-1} B$$

$$A^{-1} = \frac{1}{|A|} . \text{Adj. } A$$

$$|A| = \begin{vmatrix} 1 & 3 & 4 \\ 4 & 2 & 3 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 1 \times \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} - 3 \times \begin{vmatrix} 4 & 3 \\ 1 & 1 \end{vmatrix} + 4 \times \begin{vmatrix} 4 & 2 \\ 1 & 1 \end{vmatrix}$$

$$= (2 - 3) - 3(4 - 3) + 4(4 - 2)$$

$$= -1 - 3 + 8 = -4 + 8 = 4$$

$$\text{Adj. } A = (c_{ij})^T_{3x3}$$

$$c_{11} = (-1)^{1+1} \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = (2 - 3) = -1$$

$$c_{12} = (-1)^{1+2} \begin{vmatrix} 4 & 3 \\ 1 & 1 \end{vmatrix} = -(4 - 3) = -1$$

$$c_{13} = (-1)^{1+3} \begin{vmatrix} 4 & 2 \\ 1 & 1 \end{vmatrix} = (4 - 2) = 2$$

$$c_{21} = (-1)^{2+1} \begin{vmatrix} 3 & 4 \\ 1 & 1 \end{vmatrix} = -(3 - 4) = 1$$

$$c_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix} = (1 - 4) = -3$$

$$c_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} = -(1 - 3) = 2$$

$$c_{31} = (-1)^{3+1} \begin{vmatrix} 3 & 4 \\ 2 & 3 \end{vmatrix} = (9 - 8) = 1$$

$$c_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 4 \\ 4 & 3 \end{vmatrix} = -(3 - 16) = 13$$

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$$c_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix} = (2-12) = -10$$

$$\therefore \ (c_{ij})_{3x3} = \begin{bmatrix} -1 & -1 & 2 \\ 1 & -3 & 2 \\ 1 & 13 & -10 \end{bmatrix}$$

$$\text{Adj. } A = (c_{ij})^T = \begin{bmatrix} -1 & 1 & 1 \\ -1 & -3 & 13 \\ 2 & 2 & -10 \end{bmatrix}$$

$$A^{-1} = \frac{1}{4} \begin{bmatrix} -1 & 1 & 1 \\ -1 & -3 & 13 \\ 2 & 2 & -10 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{-1}{4} & \frac{-3}{4} & \frac{13}{4} \\ \frac{2}{4} & \frac{2}{4} & \frac{-10}{4} \end{bmatrix}$$

$$\therefore \ X = \begin{bmatrix} \frac{-1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{-1}{4} & \frac{-3}{4} & \frac{13}{4} \\ \frac{2}{4} & \frac{2}{4} & \frac{-10}{4} \end{bmatrix} \begin{bmatrix} 7 \\ 10 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-7}{4} + \frac{10}{4} + \frac{3}{4} \\ \frac{-7}{4} - \frac{30}{4} + \frac{39}{4} \\ \frac{14}{4} + \frac{20}{4} - \frac{30}{4} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{6}{4} \\ \frac{2}{4} \\ \frac{4}{4} \end{bmatrix} = \begin{bmatrix} 1\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

$\therefore \quad x = 1\frac{1}{2}; \quad y = \frac{1}{2}; \quad z = 1.$

Note: This method can be used only when $|A| \neq 0$. because, if $|A| = 0$, A^{-1} does not exist.

Space for hints

3.2 Determinants Method or Cramer's Rule

Suppose we are given n simultaneous linear equations in n variables as follows:

$$
\begin{array}{l}
a_{11} x_1 + a_{12} x_2 + \ldots\ldots + a_{1n} x_n = b_1 \\
a_{21} x_1 + a_{22} x_2 + \ldots\ldots + a_{2n} x_n = b_2 \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
a_{n1} x_1 + a_{n2} x_2 + \ldots\ldots + a_{nn} x_n = b_n
\end{array}
$$

As in the Matrix Inversion method, here also we form the coefficient matrix,

$$
A = \begin{vmatrix}
a_{11} & a_{12} & \ldots\ldots & a_{1n} \\
a_{21} & a_{22} & \ldots\ldots & a_{2n} \\
\ldots & \ldots & \ldots\ldots & \ldots \\
\ldots & \ldots & \ldots\ldots & \ldots \\
a_{n1} & a_{n2} & \ldots\ldots & a_{nn}
\end{vmatrix}_{nxn}
$$

Now, the value of $|A|$ is obtained and it is denoted by Δ.

In $|A|$, we replace the first column by the respective constants on the RHS of the given equations; for the resultant determinant, we find out the value and this value is denoted by Δ_1. That is,

$$
\Delta_1 = \begin{vmatrix}
b_1 & a_{12} & a_{13} & \ldots\ldots & a_{1n} \\
b_2 & a_{22} & a_{23} & \ldots\ldots & a_{2n} \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
b_n & a_{n2} & a_{n3} & \ldots\ldots & a_{nn}
\end{vmatrix}
$$

Similarly, we replace the second column of A by the constants and denote the resultant determinant by Δ_2. That is,

$$
\Delta_2 = \begin{vmatrix}
a_{11} & b_1 & a_{13} & \ldots\ldots & a_{1n} \\
a_{21} & b_2 & a_{23} & \ldots\ldots & a_{2n} \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
a_{n1} & b_n & a_{n3} & \ldots\ldots & a_{nn}
\end{vmatrix}
$$

Space for hints

If we replace the third column by the constants, the resultant determinant is denoted by Δ_3 and so on. When the last column is replaced, the determinant is denoted by Δ_n.

Now, according to the Cramer's rule,

$$x_1 = \frac{\Delta_1}{\Delta} \; ; \quad x_2 = \frac{\Delta_2}{\Delta} \quad \ldots\ldots\ldots \quad x_n = \frac{\Delta_n}{\Delta}$$

As in the Matrix Inversion method, in this method also, $|A| = \Delta$ should not be equal to zero. If $\Delta = 0$, we cannot apply this method.

The application of Cramer's rule can be very easily understood with the help of some examples.

Example :

Solve the following equation applying Cramer's rule.

$2x + 3y = 12$

$3x + 2y = 13$

The coefficient matrix, $A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$

$$\therefore \quad |A| = \begin{matrix} 2 & 3 \\ 3 & 2 \end{matrix} = 4 - 9 = -5$$

$|A| \neq 0$

$\therefore$ We can apply Cramer's rule.

$$\Delta = |A| = -5$$

$$\Delta_1 = \begin{vmatrix} 12 & 3 \\ 13 & 2 \end{vmatrix} = 12 \text{ x } 2 - 3 \text{ x } 13$$

$$= 24 - 39 = -15$$

$$\Delta_2 = \begin{vmatrix} 2 & 12 \\ 3 & 13 \end{vmatrix} = 2 \text{ x } 13 - 3 \text{ x } 12$$

$$= 26 - 36 = -10$$

$$x = \frac{\Delta_1}{\Delta} = \frac{-15}{-5} = 3$$

$$y = \frac{\Delta_2}{\Delta} = \frac{-10}{-5} = 2$$

Example :

Solve : $2x - 3y + z = 1$

$x + y - z = 0$

$x - 2y + z = -1$

Space for hints

The coefficient matrix is

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & -1 \\ 1 & -2 & 1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & -3 & 1 \\ 1 & 1 & -1 \\ 1 & -2 & 1 \end{vmatrix}$$

$$= 2(-1)^{1+1} \begin{vmatrix} 1 & -1 \\ -2 & 1 \end{vmatrix} - 3(-1)^{1+2} \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} + 1(-1)^{1+3} \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix}$$

$$= 2\{1x1 - (-1)(-2)\} + 3\{1x1 - (-1)(1)\} + \{1(-2) - 1x1\}$$

$$= 2\{1 - 2\} + 3\{1 + 1\} + \{-2 - 1\}$$

$$= 2(-1) + 3 x 2 - 3$$

$$= -2 + 6 - 3 = -5 + 6 = 1$$

$\therefore \quad |A| \neq 0$

$\therefore \quad x = \dfrac{\Delta_1}{\Delta} \qquad y = \dfrac{\Delta_2}{\Delta} \quad z = \dfrac{\Delta_3}{\Delta}$

$\Delta = 1$

$$\Delta_1 = \begin{vmatrix} 1 & -3 & 1 \\ 0 & 1 & -1 \\ -1 & -2 & 1 \end{vmatrix}$$

$$= 1(-1)^{1+1} \begin{vmatrix} 1 & -1 \\ -2 & 1 \end{vmatrix} + 0(-1)^{2+1} \begin{vmatrix} -3 & 1 \\ -2 & 1 \end{vmatrix} - 1(-1)^{3+1} \begin{vmatrix} -3 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= +1\{1 - (-1)(-2)\} + 0 - \{(-3)(-1) - 1x1\}$$

$$= (1 - 2) - (3 - 1)$$

$$= -1 - 2 = -3$$

$$\Delta_2 = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 1 \end{vmatrix}$$

$$= 2(-1)^{1+1} \begin{vmatrix} 0 & -1 \\ -1 & 1 \end{vmatrix} + 1(-1)^{1+2} \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} + (-1)^{1+3} \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix}$$

$$= 2\{0x1 - (-1)(-1)\} - \{1x1 - (-1) x 1\} + \{1(-1) - 0x1\}$$

$$= -2 - (1 + 1) - 1$$

$$= -2 - 2 - 1 = -5$$

$$\Delta_3 = \begin{vmatrix} 2 & -3 & 1 \\ 1 & 1 & 0 \\ 1 & -2 & -1 \end{vmatrix}$$

$$= 2(-1)^{1+1}\begin{vmatrix} 1 & 0 \\ -2 & -1 \end{vmatrix} + 1(-1)^{2+1}\begin{vmatrix} -3 & 1 \\ -2 & -1 \end{vmatrix} + 1(-1)^{3+1}\begin{vmatrix} -3 & 1 \\ 1 & 0 \end{vmatrix}$$

$$= 2\{1(-1)-(-2)\text{x}0\} - \{(-3)(-1)-(-2)\text{x}1\} + \{(-3)\text{x}0-1\text{x}1\}$$

$$= -2 - 5 - 1$$

$$= -8$$

$$\therefore \quad x = \frac{\Delta_1}{\Delta} = \frac{-3}{1} = -3$$

$$y = \frac{\Delta_2}{\Delta} = \frac{-5}{1} = -5$$

$$z = \frac{\Delta_3}{\Delta} = \frac{-8}{1} = -8$$

The solution of the given equations is

$(-3, \ -5, \ -8)$.

Example :

Solve the following equations using Cramer's rule.

$$x + 2y + 3z = 14$$
$$3x + 2y + z = 10$$
$$6x + 4y + 2z = 25$$

Coefficient matrix is

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 6 & 4 & 2 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 6 & 4 & 2 \end{vmatrix}$$

$$= (-1)^{1+1}\begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} + 2(-1)^{1+2}\begin{vmatrix} 3 & 1 \\ 6 & 2 \end{vmatrix} + 3(-1)^{1+3}\begin{vmatrix} 3 & 2 \\ 6 & 4 \end{vmatrix}$$

$$= (4-4) - 2(6-6) + 3(12-12)$$

$$= 0 - 0 + 0 = 0.$$

$\therefore$ The given equations cannot be solved using Cramer's rule.

If in the given equations, anyone of the variable is not found in any equation, its coefficient is written as zero while writing the coefficient matrix.

Space for hints

Space for hints

Example :

Solve.
$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 3 \\ 2x_1 - x_3 &= 4 \\ 4x_1 + 2x_2 + 2x_3 &= 5 \end{aligned}$$

coefficient matrix is

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & -1 \\ 4 & 2 & 2 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 0 & -1 \\ 4 & 2 & 2 \end{vmatrix}$$

$$= (-1)^{1+} \begin{vmatrix} 0 & -1 \\ 2 & 2 \end{vmatrix} + 2(-1)^{1+2} \begin{vmatrix} 2 & -1 \\ 4 & 2 \end{vmatrix} + 3(-1)^{1+3} \begin{vmatrix} 2 & 0 \\ 4 & 2 \end{vmatrix}$$

$$= [0 - (-1)(2)] - 2[4 - (-1)(4)] + 3[4 - 0]$$

$$= 2 - 2 \times 8 + 12$$

$$= 14 - 16 = -2$$

$$|A| \neq 0$$

$$\therefore x_1 = \frac{\Delta_1}{\Delta} \quad x_2 = \frac{\Delta_2}{\Delta} \quad x_3 \quad \frac{\Delta_3}{\Delta}$$

$$\Delta = -2$$

$$\Delta_1 = \begin{vmatrix} 3 & 2 & 3 \\ 4 & 0 & -1 \\ 5 & 2 & 2 \end{vmatrix}$$

$$= 3(-1)^{1+1} \begin{vmatrix} 0 & -1 \\ 2 & 2 \end{vmatrix} + 2(-1)^{1+2} \begin{vmatrix} 4 & -1 \\ 5 & 2 \end{vmatrix} + 3(-1)^{1+3} \begin{vmatrix} 4 & 0 \\ 5 & 2 \end{vmatrix}$$

$$= 3(0 + 2) - 2(8 + 5) + 3(8 - 0)$$

$$= 6 - 26 + 24 = 30 - 26 = 4$$

$$\Delta_2 = \begin{vmatrix} 1 & 3 & 3 \\ 2 & 4 & -1 \\ 4 & 5 & 2 \end{vmatrix}$$

$$= (-1)^{1+1} \begin{vmatrix} 4 & -1 \\ 5 & 2 \end{vmatrix} + 3(-1)^{1+2} \begin{vmatrix} 2 & -1 \\ 4 & 2 \end{vmatrix} + 3(-1)^{1+3} \begin{vmatrix} 2 & 4 \\ 4 & 5 \end{vmatrix}$$

$$= (8 + 5) - 3(4 + 4) + 3(10 - 16)$$

$$= 13 - 24 - 18 = 13 - 42 = -29$$

Space for hints

$$\Delta_3 = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 0 & 4 \\ 4 & 2 & 5 \end{vmatrix}$$

$$= (-1)^{1+1} \begin{vmatrix} 0 & 4 \\ 2 & 5 \end{vmatrix} + 2(-1)^{1+2} \begin{vmatrix} 2 & 4 \\ 4 & 5 \end{vmatrix} + 3(-1)^{1+3} \begin{vmatrix} 2 & 0 \\ 4 & 2 \end{vmatrix}$$

$$= (0 - 8) - 2(10 - 16) + 3(4 - 0)$$

$$= -8 + 12 + 12 = -8 + 24 = 16$$

$$\therefore x_1 = \frac{\Delta_1}{\Delta} = \frac{4}{-2} = -2$$

$$x_2 = \frac{\Delta_2}{\Delta} = \frac{-29}{-2} = \frac{29}{2}$$

$$x_3 = \frac{\Delta_3}{\Delta} = \frac{16}{-2} = -8$$

Example :

Solve.

$$x + y + z = 6$$
$$2x - y + z = 3$$
$$3x + 2y + 3z = 16$$

Coefficient matrix of the given equations is

$$A = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 3 & 2 & 3 \end{vmatrix}$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 3 & 2 & 3 \end{vmatrix}$$

$$= 1 \times [(-1) \times 3 - 1 \times 2] - 1 [2 \times 3 - 1 \times 3] + 1 [2 \times 2 - (-1) \times 3]$$

$$= (-3 - 2) - (6 - 3) + (4 + 3)$$

$$= -5 - 3 + 7 = -8 + 7 = -1 \neq 0$$

$\therefore$ Using Cramer's rule we can solve the given equations as

$$\therefore \quad x = \frac{\Delta_1}{\Delta}, \quad y = \frac{\Delta_2}{\Delta} \quad \text{and} \quad z = \frac{\Delta_3}{\Delta}$$

$$\Delta = |A| = -1$$

$$\Delta_1 = \begin{vmatrix} 6 & 1 & 1 \\ 3 & -1 & 1 \\ 16 & 2 & 3 \end{vmatrix}$$

$$= 6 \times [(-1) \times 3 - 1 \times 2] - 1 \times [3 \times 3 - 1 \times 16] + 1 \times [3 \times 2 - (-1) \times 16]$$

$$= 6\,(-3-2) \; - \; (9-16) \; + \; (6+16)$$

$$= 6 \text{ x } (-5) \; - \; (-7) \; + \; (22)$$

$$= -30 + 7 + 22 \; = \; -30 + 29 \; = \; -1$$

$$\Delta_2 = \begin{vmatrix} 1 & 6 & 1 \\ 2 & 3 & 1 \\ 3 & 16 & \text{[illegible]} \end{vmatrix}$$

$$= 1 \text{ x } [3 \text{ x } 3 - 1 \text{ x } \text{[illegible]} \; - \; 6\,[2 \text{ x } 3 - 1 \text{ x } 3] \; + \; [2 \text{ x } 16 - 3 \text{ x } 3]$$

$$= (9-16) \; - \; 6\,(6-3) \; + (32-9)$$

$$= -7 - 18 + 23$$

$$= -25 + 23 \; = \; -2$$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 6 \\ 2 & -1 & 3 \\ 3 & 2 & 16 \end{vmatrix}$$

$$= 1 \text{ x } [(-1) \text{ x } 16 - 3 \text{ x } 2] \; - \; 1 \text{ x } [2 \text{ x } 16 - 3 \text{ x } 3] \; + \; 6 \text{ x } [2 \text{ x } 2 - (-1) \text{ x } 3]$$

$$= (-16-6) \; - \; (32-9) \; + \; 6(4+3)$$

$$= -22 - 23 + 42$$

$$= -45 + 42 \; = \; -3$$

$$\therefore \quad x = \frac{\Delta_1}{\Delta} = \frac{-1}{-1} = 1$$

$$y = \frac{\Delta_2}{\Delta} = \frac{-2}{-1} = 2$$

$$z = \frac{\Delta_3}{\Delta} = \frac{-3}{-1} = 3$$

Exercise :

Solve the following equations by the matrix method and check your answer by the determinants method.

1. $3x + y = 1$
   $5x + 2y = 3$

2. $2x_1 - 3x_2 = 7$
   $x_1 + 4x_2 = 1$

3. $x_1 - x_2 + 2x_3 = 1$
   $2x_1 + 2x_2 = 3$
   $3x_1 + x_2 + 3x_3 = 7$

Space for hints

4.
$$2x_1 + x_2 + x_3 + x_4 = 2$$
$$3x_1 - x_2 + x_3 - x_4 = 2$$
$$x_1 + 2x_2 - x_3 + x_4 = 1$$
$$6x_1 + 2x_2 + x_3 + x_4 = 5$$

4. Answers to Check Your Progress Questions

1.	Refer	1	6.	Refer	1.3
2.	Refer	1.3	7.	Refer	1.3
3.	Refer	1.3	8.	Refer	1.3
4.	Refer	1.3	9.	Refer	1.3
5.	Refer	1.3	10.	Refer	3.1

5. Model questions for guidance :

10 Marks Questions (One Page Answer) :

1) Define the term matrix. Explain the types of matrices with example.

2)
$$A = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 4 & 7 & 6 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

Find AB.

3) Solve by Cramer's rule.
$$5x - 2y + 3z = 16$$
$$2x + 3y - 5z = 2$$
$$4x - 5y + 6z = 7$$

4) Solve by Cramer's rule.
$$3x - 2y + 4z = 19$$
$$6x + 2y - z = 37$$
$$x + 2y + 3z = 10$$

20 Marks Questions (Three Page Answer) :

1) Obtain the solution by Cramer's rule and verify it for the system.
$$2x + 4y - z = 15$$
$$x - 2y + 2z = -2$$
$$3x - y + 3z = 6$$

Space for hints

2)

$$A = \begin{bmatrix} 2 & -3 & 0 \\ 3 & 1 & -2 \\ -1 & 0 & -4 \end{bmatrix} \quad \text{Find } A^{-1}$$

3)

$$\text{If } A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 0 & -3 \\ 3 & 4 & 0 \end{bmatrix} \quad \text{Find } A^{-1}$$

4) Solve the following equations by using Cramer's rule.

$$2x_1 + 3x_2 - x_3 = 9$$
$$x_1 + x_2 + x_3 = 9$$
$$3x_1 - x_2 - x_3 = -1$$

5) Solve the following simulaneous equations by matrix method.

$$3x + 5y + z = 36$$
$$x + 2y + 4z = 42$$
$$4x + 3y + 2z = 28$$

Space for hints

UNIT – 10

DIFFERENTIAL CALCULUS - ITS APPLICATION IN ECONOMICS

Introduction :

In Unit - 8, we already saw the concept of Function. In a function, the rate of change of the dependent variable when there is a minute change in the independent is called 'derivative'. The method of arriving at the derivative of given function is dealt with in the branch of Mathematics, called Differential Calculus. Derivative, also known as differential coefficient, finds wide application in Economics. Therefore, the meaning, the method of finding it for important functions and its applications in Econoics are all described in this Unit - 10.

Unit Objectives :

After studying this Unit, you would be able to understand

* the meaning and the method of computing the derivative for some important functions given
* the various uses of derivative particularly in Economics.

Unit Structure

1. Differentiation and Differential Coefficient - Meaning
2. Symbolic Representation of Derivative
3. Differentiation of some elementary functions
4. Rules of Diffentiation
5. Successive Differentiation
6. Maximum and Minimum values of a function of single variable
7. Applications of Functions and Diagrams in Economic Theory
8. Elasticity Concepts in Economics
9. Answers to Check Your Progress Questions
10. Model questions for guidance

1. Differentiation and Differential Coefficient - Meaning

When two variables are given, if one variable depends on the second variable, the nature of relationship between the two variables can be expressed in the form of an equation. This equation is called a function. The variable written on the RHS of the equation is called independent variable and the variable written on the LHS of the equation is called dependent variable. When the independent variable takes different values, the dependent variable also changes in its value correspondingly. The average rate of change in the value of the dependent variable corresponding to a

Space for hints

change in the value of the independent variable is called derivative or differential coefficient and the process of finding.

Let us assume that $y = f(x)$ is the given function. Here, x is the independent variable and y is the dependent variable.

Suppose x takes the value x_1 at first. When $x = x_1$, the value of y will be $f(x_1)$.

Also suppose that the value of x changes from x_1 to x_2. Now the value of y will change from $f(x_1)$ to $f(x_2)$. That is, when the change in the value of x is $(x_2 - x_1)$ the change in y is $f(x_2) - f(x_1)$. Therefore, average rate of change in the variable y corresponding to the change in x is $\frac{f(x_2) - f(x_1)}{x_2 - x_1}$. This concept can be easily understood with the help of an example.

Consider the function $y = x^2$

Suppose x takes the value 1. When $x = 1$, the value of $y = (1)^2 = 1$.

Suppose the value of x increases from 1 to 3. When $x = 3$, $y = (3)^2 = 9$. Thus, when x increases from 1 to 3, y increases from 1 to 9. Now, the average rate of change in the value of y when x changes from 1 to 3 = $\frac{9-1}{3-1} = \frac{8}{2} = 4$.

Suppose the value of x changes from 1 to (1+h) instead of 1 to 3.

when $x = (1+h)$, $y = (1+h)^2 = 1 + 2h + h^2$.

$\therefore$ Average rate of change in the value of y when x changes from 1 to (1+h) is

$$\frac{(1 + 2h + h^2) - 1}{(1 + h) - 1} = \frac{2h + h^2}{h} = 2 + h$$

When the value of h is infinitesimally small (that is, very close to zero), the average rate of change in the value of y will be very close to 2. This value of $y = 2$ is called 'the instantaneous rate of change in the value of y when $x = 1$'.

When x takes a particular value of x the instaneous rate of change in y is called the 'derivative' or 'differential coefficient of the given function'.

2. Symbolic Representation of Derivative

Check your Progress

1. What is meant by derivative?

Generally, a small change in x is denoted by Dx and the corresponding change in y is denoted by (the symbol D is read as delta). When Dx tends to zero the value of the ratio $\frac{\Delta y}{\Delta x}$ is called the differential coefficient of y with respect to x. That is,

the limiting value of the ratio $\frac{\Delta y}{\Delta x}$ is called the differential coefficient.

This limiting value is usually denoted by $\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$

This is read as Limit Δx tends to zero $\frac{\Delta y}{\Delta x}$

Derivative or differential coefficient is generally denoted by $\frac{dy}{dx}$ or $f'(x)$ (read as f dashed x).

We can give the definition of derivative as follows.

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + Dx) - f(x)}{\Delta x}$$

3. Differentiation of some elementary functions

Here we consider the differentiation of the functions nar ıy,

(1) $y = x^n$

(2) $y = (x + a)^n$

(3) $y = \log x$

(4) $y = e^x$

As derivations of formulae are not included in your syllabus, we directly give below the formula to get the derivative or differential coefficient for each of the functions given above.

Function	Derivative
$y = x^n$	$\frac{dy}{dx} = n\ x^{n-1}$
$y = (x+a)^n$	$\frac{dy}{dx} = n\ (x+a)^{n-1}$
$y = \log x$	$\frac{dy}{dx} = \frac{1}{x}$
$y = e^x$	$\frac{dy}{dx} = e^x$

Example 1:

If $y = x^5$ find the value of $\frac{dy}{dx}$

Space for hints

Check your Progress

2. Give the symbolic representation of derivative.

Space for hints

If $y = x^n$, $\frac{dy}{dx} = n\,x^{n-1}$

$y = x^5$

$\therefore\ n = 5 \qquad n - 1 = 4$

$\therefore\ \frac{dy}{dx} = 5\,x^4$

Example 2:

$f(x) = x^7$. Find $f'(x)$

$f(x) = x^7$

$\therefore\ n = 7 \qquad n - 1 = 6$

$\therefore\ \frac{dy}{dx} = f'(x) = 7x^6$.

Example 3:

$y = \frac{1}{x}$ Find $\frac{dy}{dx}$.

$y = \frac{1}{x}$

First of all the given function must be rewritten in the form of $y = x^n$ as follows.

$y = \frac{1}{x} = x^{-1}$

$\therefore\ \frac{dy}{dx} = (-1)x^{-1-1} = -x^{-2} = \frac{-1}{x^2}$

Example 4:

$y = x^{1/3}$. Find $\frac{dy}{dx}$

$y = x^{1/3}$

$$\therefore\ \frac{dy}{dx} = \frac{1}{3}x^{\frac{1}{3}-1} = \frac{1}{3}x^{\frac{1-3}{3}} = \frac{1}{3}x^{\frac{-2}{3}}$$

$$= \frac{1}{3x^{\frac{2}{3}}}$$

Example 5:

If $y = (x+2)^8$, find $\frac{dy}{dx}$.

Space for hints

If $y = (x+a)^n$, $\frac{dy}{dx} = n(x+a)^{n-1}$

Given $y = (x+2)^8$. $\therefore \frac{dy}{dx} = 8(x+2)^{8-1}$

$$= 8(x+2)^7$$

Example 6:

$$y = \frac{1}{(x+3)} \quad \text{Find } \frac{dy}{dx}.$$

$$y = \frac{1}{(x+3)^5} = (x+3)^{-5}$$

$$\therefore \frac{dy}{dx} = (-5)(x+3)^{-5-1} = -5(x+3)^{-6}$$

$$= \frac{-5}{(x+3)^6}$$

Example 7:

$$y = \frac{1}{\sqrt{x+7}} \quad \text{Find } \frac{dy}{dx}.$$

$$y = \frac{1}{\sqrt{x+7}} = \frac{1}{(x+7)^{1/2}} = (x+7)^{-1/2}$$

$$\therefore \frac{dy}{dx} = \left(-\frac{1}{2}\right)(x+7)^{-1/2-1} = \frac{1}{2}(x+7)^{-3/2}$$

$$= \frac{-1}{2(x+7)^{3/2}}$$

4. Rules of Differentiation

4.1 Derivative of a constant is equal to zero. That is, if $y = f(x) = a$ where a is constant, then $\frac{dy}{dx} = f'(x) = 0$.

4.2 If y is the product of a constant = a and a function of x = f(x) then the derivative will also be the product of the constant a and the derivative of the function, f'(x). That is,

If $y = a\, f(x)$,

$$\frac{dy}{dx} = a\, f'(x)$$

4.3 If y is the sum of two functions, f(x), and g(x), then the derivative of y is also the sum of the derivatives of the two functions.

That is, if $y = f(x) + g(x)$ then $\frac{dy}{dx} = f'(x) + g'(x)$

Space for hints

4.4 If y is the difference of two functions f(x) and g(x), then the derivative will also be the difference of the derivatives of the two functions. That is, if $y = f(x) - g(x)$,

$$\frac{dy}{dx} = f'(x) - g'(x)$$

4.5 Product rule :

If y is the product of two functions, then the derivative will be the sum of the product of each function with the derivative of the other's.

That is, if $y = f(x) \cdot g(x)$

$$\frac{dy}{dx} = f(x) \cdot g'(x) + g(x) \cdot f'(x)$$

If we put $f(x) = u$ and $g(x) = v$, then $y = uv$ and

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

4.6 Quotient rule :

Derivative of the quotient of two functions is equal to [Denominator x derivative of numerator – Numerator x derivative of denominator] , (Denominator)^2

Suppose, $y = \frac{f(x)}{g(x)}$,

Put $f(x) = u$ and $g(x) = v$

Now, $y = \frac{u}{v}$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

4.7 Chain rule :

If y is a function of the variable u and u is again a function of the variable x, then the derivative of y is given as follows.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

That is, if $y = f(u)$ and $u = g(x)$,

then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

All the above seven rules may be given in compact form in a table as follows:

Function type	Rule
$y = a$	$\frac{dy}{dx} = 0$
$y = a\, f(x)$	$\frac{dy}{dx} = a\,.\, f'(x)$
$y = f(x) + g(x)$	$\frac{dy}{dx} = f'(x) + g'(x)$
$y = f(x) - g(x)$	$\frac{dy}{dx} = f'(x) - g'(x)$
$y = uv$	$\frac{dy}{dx} = u\,.\,\frac{dv}{dx} + v\,.\,\frac{du}{dx}$
$y = \frac{u}{v}$	$\frac{dy}{dx} = \frac{v.\frac{du}{dx} - u.\frac{dv}{dx}}{v^2}$
$y = f(u),\ u = g(x)$	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

Example 1:

Find the derivatives of the following functions.

(i) $y = 8$

(ii) $y = 5x^7$

(iii) $y = x^2 + 10$

(iv) $y = 3x^4 - 6x$

(i) $y = 8$, a constant.

$$\therefore \quad \frac{dy}{dx} = 0$$

(ii) $y = 5x^7$

$$\therefore \quad \frac{dy}{dx} = 5 \times 7x^{7-1} = 35\, x^6$$

(iii) $y = x^2 + 10$

$$\frac{dy}{dx} = 2 \times x^{2-1} + 0$$

$$= 2x$$

Space for hints

Space for hints

(iv) $y = 3x^4 - 6x$

$$= 3 \times 4x^{4-1} - 6 \times 1x^{1-1}$$

$$= 12x^3 - 6x^0 = 12x^3 - 6$$

Example 2

$y = 5(x+3)^4$. Find $\frac{dy}{dx}$.

$y = 5\,(x+3)^4$

$$\therefore \frac{dy}{dx} = 5 \times 4(x+3)^{4-1}$$

$$= 20(x+3)^3$$

Example 3

$y = 11 - (x+7)^2$. Find $\frac{dy}{dx}$.

$y = 11 - (x+7)^2$

$$\therefore \frac{dy}{dx} = 0 - 2(x+7)^{2-1}$$

$$= -2(x+7)^1$$

$$= -2(x+7)$$

$$= -2x - 14.$$

Example 4

$y = (x+1)^{10} + 8x^3$. Find $\frac{dy}{dx}$.

$y = (x+1)^{10} + 8x^3$

$$\therefore \frac{dy}{dx} = 10(x+1)^{10-1} + 8 \times 3x^{3-1}$$

$$= 10(x+1)^9 + 24x^2$$

Example 5

$y = 5 \log x + 7e^x$. Find $\frac{dy}{dx}$.

$y = 5 \log x + 7e^x$

$$\therefore \frac{dy}{dx} = 5 \times \frac{1}{x} + 7e^x$$

$$= \frac{5}{x} + 7e^x$$

Space for hints

Example 6

$y = 8 \log x + e^x + x^5$. Find $\frac{dy}{dx}$.

$y = 8 \log x + e^x + x^5$

$$\therefore \frac{dy}{dx} = 8 \times \frac{1}{x} + e^x + 5 \text{ X } x^{5-1}$$

$$= \frac{8}{x} + e^x + 5x^4$$

Example 7

$y = (7x + 9)(8x + 3)$. Find $\frac{dy}{dx}$.

Here y is a product of two functions $(7x + 9)$ and $(8x + 3)$

Put $(7x + 9) = u$

$(8x + 3) = v$

$\therefore$ $y = uv$.

$$\text{Now, } \frac{dy}{dx} = u.\frac{dv}{dx} + v.\frac{du}{dx}$$

$$u = (7x + 9) \quad \therefore \frac{du}{dx} = 7$$

$$v = (8x + 3) \quad \therefore \frac{dv}{dx} = 8$$

$$\therefore \frac{dy}{dx} = (7x + 9) \times 8 + (8x + 3) \times 7$$

$$= 56x + 72 + 56x + 21$$

$$= 112x + 93$$

Example 8

Find the derivative of the function $(3x^3 - 7x - 9)(5x^2 - 11x + 8)$.

Let $y = (3x^3 - 7x - 9)(5x^2 - 11x + 8)$

Put $u = 3x^3 - 7x - 9$

$$\therefore \frac{du}{dx} = 9x^2 - 7 - 0 = 9x^2 - 7$$

Space for hints

Put $v = 5x^2 - 11x + 8$

$$\therefore \frac{dv}{dx} = 10x - 11 + 0 = 10x - 11$$

Now, $y = uv$

$$\therefore \frac{dy}{dx} = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$$

$$= (3x^3 - 7x - 9)(10x - 11) + (5x^2 - 11x + 8)(9x^2 - 7)$$

$$= [30x^4 - 70x^2 - 90x - 33x^3 + 77x + 99] + [45x^4 - 99x^3 + 72x^2 - 35x^2 + 77x - 56]$$

$$= 30x^4 + 45x^4 - 33x^3 - 99x^3 - 70x^2 + 72x^2 - 90x + 77x + 77x + 99 - 56$$

$$= 75x^4 - 132x^3 + 2x^2 + 64x + 43$$

Example 9

Differentiate the following function.

$$y = \frac{3x - 1}{4x + 5}$$

Put $u = 3x - 1$

$v = 4x + 5$

$$\therefore y = \frac{u}{v}$$

$$\frac{dy}{dx} = \frac{v \dfrac{du}{dx} - u \dfrac{dv}{dx}}{v^2}$$

$$\frac{du}{dx} = 3 \times 1 - 0 = 3$$

$$\frac{dv}{dx} = 4 \times 1 + 0 = 4$$

$$\therefore \frac{dy}{dx} = \frac{(4x + 5) \times 3 - (3x - 1) \times 4}{(4x + 5)^2} = \frac{12x + 15 - 12x + 4}{(4x + 5)^2}$$

$$= \frac{19}{(4x+5)^2}$$

Example 10

Differentiate $\dfrac{4}{(3x + 1)}$

Space for hints

Let $y = \dfrac{4}{(3x + 1)}$

Put $u = 4 \quad v = 3x + 1$

$$\frac{du}{dx} = 0 \quad \frac{dv}{dx} = 3$$

$$\therefore \frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

$$= \frac{(3x + 1) \times 0 - 4 \times 3}{(3x + 1)^2}$$

$$= \frac{0 - 12}{(3x + 1)^2}$$

$$= \frac{-12}{(3x + 1)^2}$$

Example 11

If $y = \log 2x$. Find $\dfrac{dy}{dx}$

$y = \log 2x.$

Put $2x = u$

$\therefore y = \log u$

$$\frac{dy}{du} = \frac{1}{u}$$

$$\frac{du}{dx} = 2 \times x^{1-1} = 2$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{u} \times 2$$

$$= \frac{1}{2x} \times 2$$

$$= \frac{1}{x}$$

Example 12

$y = e^{-x}$. Find $\dfrac{dy}{dx}$

$y = e^{-x}$

Space for hints

Put $u = -x$,

$\therefore y = e^u$

$\frac{dy}{du} = e^u$

$\frac{du}{dx} = -1 \times x^{1-1} = -x^0 = -1.$

$\therefore \frac{dy}{dx} = \frac{dv}{du} \cdot \frac{du}{dx}$

$= e^u (-1)$

$= e^{-x} (-1)$

$= -e^{-x}$

Example 13

$y = (x^2 - e^{2x})^5$. Find $\frac{dy}{dx}$

$y = (x^2 - e^{2x})^5$

Put $u = x^2 - e^{2x}$

$\therefore y = u^5$

$\frac{dy}{du} = 5 \times u^{5-1} = 5u^4 = 5(x^2 - e^{2x})^4$

$\frac{du}{dx} = 2x - e^{2x} \times 2$

$= 2x - 2e^{2x} = 2(x - e^{2x})$

$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{dx}{dx}$

$= 5(x^2 - e^{2x})^4 \ 2(x - e^{2x})$

$= 10(x^2 - e^{2x})^4 \ (x - e^{2x})$

Example 14

Differentiate the following.

(a) $2x^{½} + \frac{3}{x} + 4$

(b) $(x + 1)(x^2 - 5)$

(c) $\frac{x^2 - 5}{x + 4}$

Solution

Space for hints

(a) Let $y = 2x^{½} + \dfrac{3}{x} + 4.$

$$\frac{dy}{dx} = 2 \times ½ \times x^{½-1} + 3 \times (-1) \times \frac{1}{x^2} + 0$$

$$= x^{-½} - \frac{3}{x^2}$$

$$= \frac{1}{x^{½}} - \frac{3}{x^2}$$

(b) Let $y = (x + 1)(x^2 - 5)$

Put $u = (x+1) \quad \therefore \quad \dfrac{du}{dx} = 1 + 0 = 1$

$v = (x^2 - 5) \qquad \dfrac{dv}{dx} = 2x - 0 = 2x$

$\therefore \quad y = uv$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$= (x + 1) \times 2x + (x^2 - 5) \times 1$$

$$= 2x^2 + 2x + x^2 - 5$$

$$= 3x^2 + 2x - 5$$

(c) Let $y = \dfrac{x^2 - 5}{x + 4}$

Put $u = x^2 - 5 \quad \therefore \quad \dfrac{du}{dx} = 2x - 0 = 2x$

$v = x + 4 \quad \therefore \quad \dfrac{dv}{dx} = 1 + 0 = 1$

$y = \dfrac{u}{v}$

$$\therefore \quad \frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

$$= \frac{(x + 4)\,2x - (x^2 - 5) \times 1}{(x + 4)^2}$$

Space for hints

$$= \frac{2x^2 + 8x - x^2 + 5}{(x+4)^2}$$

$$= \frac{x^2 + 8x + 5}{(x+4)^2}$$

Example 15

Differentiate the following with respect to x.

(i) $\dfrac{x^2 + x + 1}{x^2 - x + 1}$

(ii) $\dfrac{(x^2 + 1)}{(2x + 3)^4}$

(iii) $\dfrac{1}{\sqrt{1-x}}$

Solution

(i) Let $y = \dfrac{x^2 + x + 1}{x^2 - x + 1}$

Put $u = x^2 + x + 1 \quad \therefore \dfrac{du}{dx} = 2x + 1$

$v = x^2 - x + 1 \quad \therefore \dfrac{dv}{dx} = 2x - 1$

$$y = \frac{u}{v}$$

$$\therefore \frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

$$= \frac{(x^2 - x + 1)(2x + 1) - (x^2 + x + 1)(2x - 1)}{(x^2 - x + 1)}$$

$$= \frac{[2x^3 - 2x^2 + 2x + x^2 - x + 1] - [2x^3 + 2x^2 + 2x - x^2 - x - 1]}{(x^2 - x + 1)^2}$$

$$= \frac{2x^3 - 2x^3 - 2x^2 + x^2 - 2x^2 + x^2 + 2x - x - 2x + x + 1 + 1}{(x^2 - x + 1)^2}$$

$$= \frac{-2x^2 + 2}{(x^2 - x + 1)^2}$$

$$= \frac{2(1 - x^2)}{(x^2 - x + 1)^2}$$

Space for hints

(ii) Let $y = \dfrac{(x^2+1)}{(2x+3)^4}$

Put $u = (x^2+1) \quad \therefore \dfrac{du}{dx} = 2x$

$$v = (2x+3)^4$$

$$\frac{dv}{dx} = 4(2x+3)^3 \times 2$$

$$= 8(2x+3)^3$$

$$y = \frac{u}{v}$$

$$\therefore \frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

$$= \frac{(2x+3)^4 \times 2x - (x^2+1) \times 8(2x+3)^3}{[(2x+3)^4]^2}$$

$$= \frac{2x(2x+3)^4 - 8(2x+3)^3(x^2+1)}{(2x+3)^8}$$

$$= \frac{(2x+3)^3[2x(2x+3) - 8(x^2+1)]}{(2x+3)^8}$$

$$= \frac{2x(2x+3) - 8(x^2+1)}{(2x+3)^5}$$

$$= \frac{4x^2 + 6x - 8x^2 - 8}{(2x+3)^5} = \frac{-4x^2 + 6x - 8}{(2x+3)^5}$$

(iii) Let $y = \dfrac{1}{\sqrt{1-x}}$

$$= \frac{1}{(1-x)^{1/2}}$$

$$= (1-x)^{-1/2}$$

$$\frac{dy}{dx} = (-1/2)\,(1-x)^{-1/2-1}\,(0-1)$$

$$= (-1/2)\,(-1)\,(1-x)^{-3/2}$$

$$= \frac{1}{2(1-x)^{3/2}}$$

Space for hints

Example 16

Differentiate the following :

(a) $3x^2 - \dfrac{5}{x} + \log 3x$

(b) $\dfrac{x^2 - 2x + 3}{x^2 + 4x - 5}$

(c) $e^{2x+5} \log (4x - 3)$

Solution

(a) Let $y = 3x^2 - \dfrac{5}{x} + \log^{3x}$

$$\frac{dy}{dx} = 3 \times 2x - \frac{5(-1)}{x^2} + \frac{1}{3x} \times 3$$

$$= 6x + \frac{5}{x^2} + \frac{1}{x}$$

(b) Let $y = \dfrac{x^2 - 2x + 3}{x^2 + 4x - 5}$

Put $u = x^2 - 2x + 3$

$v = x^2 + 4x - 5$

$$\frac{du}{dx} = 2x - 2$$

$$\frac{dv}{dx} = 2x + 4$$

$$y = \frac{u}{v}$$

$$\therefore \frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

$$= \frac{(x^2 + 4x - 5)(2x - 2) - (x^2 - 2x + 3)(2x + 4)}{(x^2 + 4x - 5)^2}$$

$(x^2 + 4x - 5)(2x - 2) = (2x^3 + 8x^2 - 10x - 2x^2 - 8x + 10)$ I

$(x^2 - 2x + 3)(2x + 4) = (2x^3 - 4x^2 + 6x + 4x^2 - 8x + 12)$ II

$\mathrm{I} - \mathrm{II} = 12x^2 - 16x - 6x^2 + 0 - 2$

$= 6x^2 - 16x - 2$

$$\therefore \frac{dy}{dx} = \frac{6x^2 - 16x - 2}{(x^2 + 4x - 5)^2}$$

(c) Let $y = e^{2x+5} \log^{(4x-3)}$

Put $u = e^{2x+5}$

$v = \log^{(4x-3)}$

$\therefore \; y = u \cdot v$

$$\frac{dy}{dx} = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$$

$$\frac{du}{dx} = e^{2x+5} \; (2+0)$$

$$= 2\, e^{2x+5}$$

$$\frac{dv}{dx} = \frac{1}{(4x-3)} \times (4-0)$$

$$= \frac{4}{(4x-3)}$$

$$\therefore \frac{dy}{dx} = e^{2x+5} \times \frac{4}{(4x-3)} + \log^{(4x-3)} \times 2\, e^{2x+5}$$

$$= \frac{4\, e^{2x+5}}{(4x-3)} + 2\, e^{2x+5} \cdot \log^{(4x-3)}$$

5. Successive Differentiation

We saw earlier the method of finding out the derivative of a function, $y = f(x)$. For example, if

$$y = 3x^4 - 4x^2 + 7x + 2$$

$$\frac{dy}{dx} = 3 \text{ X } 4x^3 - 4 \times 2x + 7 + 0$$

$$= 12x^3 - 8x + 7$$

Here, it is to be noted that $\frac{dy}{dx}$ is also function of x, Therefore we can differentiate this function and find the derivative as follows :

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = 12 \times 3x^2 - 8 \times 1 + 0$$

$$= 36x^2 - 8$$

$\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is called the second order derivative of the given function, $y = f(x)$.

This second order derivative is denoted by $\frac{d^2y}{dx^2}$

$$\therefore \frac{d^2y}{dx^2} = 36x^2 - 8$$

Space for hints

Space for hints

Again it is to be noted that $\frac{d^2y}{dx^2}$ is also a function of x. Therefore, we can differentiate this function also. That is, we can find out the derivative, $\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$. In our example,

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = 36 \times 2x - 0$$

$$= 72x$$

$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$ is called the third order derivative of the given function and it is denoted by $\frac{d^3y}{dx^3}$

In the same way we can go on differentiate the given function. Generally, the n^{th} order derivative of the given function is denoted by $\frac{d^ny}{dx^n}$

Example 1:

Find the first three derivatives of the following functions.

(i) $y = x^9$

(ii) $y = x^2 + 7 \log x$

(iii) $y = e^x$

(iv) $y = (x + 3)^2$

(i) $y = x^9$

$$\therefore \frac{dy}{dx} = 9x^8$$

$$\frac{d^2y}{dx^2} = 9 \times 8x^7 = 72x^7,$$

$$\frac{d^3y}{dx^3} = 72 \times 7x^6 = 504x^6$$

(ii) $y = x^2 + 7 \log x$

$$\therefore \frac{dy}{dx} = 2x + \frac{7}{x}$$

$$\frac{d^2y}{dx^2} = 2 + \frac{7(-1)}{x^2} = 2 - \frac{7}{x^2}$$

$$\frac{d^3y}{dx^3} = 0 - \frac{7(-2)}{x^3} = \frac{14}{x^3}$$

Space for hints

(iii) $y = e^x$

$$\therefore \frac{dy}{dx} = e^x$$

$$\frac{d^2y}{dx^2} = e^x$$

$$\frac{d^3y}{dx^3} = e^x$$

(iv) $y = (x + 3)^2$

$$\therefore \frac{dy}{dx} = 2(x + 3)$$

$$\frac{d^2y}{dx^2} = 2(1 + 0)$$

$$\frac{d^3y}{dx^3} = 0$$

Example 2:

If $y = (ax - bx^2)$ prove that

$$x^2 \frac{d^2y}{dx^2} + 2y = 2x \frac{dy}{dx}$$

$y = (ax - bx^2)$

$$\therefore \frac{dy}{dx} = a - 2bx$$

$$\frac{d^2y}{dx^2} = 0 - 2b = -2b$$

We have to prove that

$$x^2 \frac{d^2y}{dx^2} + 2y = 2x \frac{dy}{dx}$$

First let us take up the LHS and substitute for $\frac{d^2y}{dx^2}$ and y.

$$x^2 \frac{d^2y}{dx^2} + 2y$$

$$= x^2(-2b) + 2(ax - bx^2)$$

$$= -2bx^2 + 2ax - 2bx^2$$

$$= -4bx^2 + 2ax.$$

Space for hints

Now let us take up the RHS and substitute for $\frac{dy}{dx}$.

$$2x\frac{dy}{dx} = 2x(a-2bx)$$

$$= 2ax - 4bx^2$$

It is to be noted that LHS = RHS.

Hence the required result is proved.

Example 3:

$y = (x^2 + a^2)^2$. Prove that,

$$(x^2 + a^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = 0$$

$y = (x^2 + a^2)^2$

$\therefore$ we need $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

To find out $\frac{dy}{dx}$, first we put $x^2 + a^2 = u$.

$\therefore\ y = u^2$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$= (2u)(2x+0)$$

$$= 2(x^2 + a^2)\,2x$$

$$= 4x(x^2 + a^2)$$

$$= 4x^3 + 4a^2x$$

$$\frac{d^2y}{dx^2} = 4 \text{ X } 3x^2 - 4a^2 \times 1$$

$$= 12x^2 + 4a^2$$

Now let us take up the L H S of the result to be proved.

$$(x^2 + a^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y$$

$$= (x^2 + a^2)(12x^2 + 4a^2) - 2x(4x^3 + 4a^2x) - 4(x^2 + a^2)^2$$

$= 12x^4 + 12a^2x^2 + 4a^2x^2 + 4a^4 - 8x^4 - 8a^2 x^2 - 4(x^4 + 2a^2x^2 + a^4)$

$= 12x^4 - 8x^4 + 16a^2 x^2 - 8a^2 x^2 + 4a^4 - 4x^4 - 8a^2 x^2 - 4a^4$

$= 12x^4 - 12x^4 + 16a^2 x^2 - 16a^2 x^2 + 4a^4 - 4a^4$

$= 0$

Hence the result is proved.

Space for hints

Exercise :

1. Find $\frac{dy}{dx}$ from the following functions.

(i) $y = x^7 + 7x^6 + 3x^2 + 15$

(ii) $y = (x - 1)(x - 2)$

(iii) $y = ax^5 + bx^7 + 8$

(iv) $y = 8 \log 3x$

(v) $y = x^2 \log x + x^3 e^x$

(vi) $y = \dfrac{x - 2}{x + 3}$

(vii) $y = \sqrt{x}(1 + x)$

(viii) $y = (x^3 + 1)(x^5 - 2)$

(ix) $y = (x^2 + 1)^{5/2}$

(x) $y = e^{3x} \log(2x + 5)$

(xi) $y = \dfrac{1}{\sqrt{x^2+3x+2}}$

(xii) $y = (\log x)^3$.

2. Find the first three derivatives from the following functions.

(i) $y = (x + 2)^7$

Space for hints

(ii) $y = ae^x + be^{-x}$

(iii) $y = \log(x^2 + x + 1)$

(iv) $y = 4x^3 + 8x^2 + 16x - 9$.

6. Maximum and minimum values of a function of single variable :

An important application of derivatives is that they provide a method of finding out the maximum or minimum values of a function. When a given function $y = f(x)$ reaches its maximum or minimum value at a point say, $x = a$, the following two conditions will be satisfied :

(i) $\frac{dy}{dx} = 0$, at $x = a$. That is, $f'(a) = 0$.

(ii) The value of the expression $\frac{d^2y}{dx^2}$ will be negative for $x = a$, when y reaches its maximum; the value of the expression $\frac{d^2y}{dx^2}$ will be positive for $x = a$, when y reaches its minimum. That is, $f''(a) < 0$ for maximum value of y and $f'(a) > 0$ for minium value of y.

The maximum and minimum values together are usually termed as **'extreme values'** of a function. To find out the extreme values of a function, the following steps are followed :

Step by step procedure :

(i) Find the first derivative, $\frac{dy}{dx}$ from the given equation, $y = f(x)$.

(ii) Equate to zero and solve the equation. That is, find the value (or values) of x from the

equation, $\frac{dy}{dx} = 0$.

(iii) Find the second order derivative, $\frac{d^2y}{dx^2}$.

In this expression, put each and every solution of x obtained in the second step ; if we get a negative value for $\frac{d^2y}{dx^2}$, the respective solution represents a point at which y reaches its maximum value ; if we get a positive vaue for $\frac{d^2y}{dx^2}$, the respective solution represents a point at which y reaches its minimum value ; if $\frac{d^2y}{dx^2} = 0$, the point represents neither a maximum nor a minimum value and at such a point the function is said to reach a stationery value.

Space for hints

Example 1:

$y = 2x^3 - 6x + 1$. Find the points at which the function reaches its maximum and/ or minimum.

$$y = 2x^3 - 6x + 1$$

$$\therefore \frac{dy}{dx} = 2 \times 3x^2 - 6 + 0 = 6x^2 - 6$$

Equating $\frac{dy}{dx}$ to zero, we get

$$6x^2 - 6 = 0$$

$$6(x^2 - 1) = 0$$

$$x^2 - 1 = 0$$

$$x^2 = 1$$

$$\therefore \quad x = +1 \text{ or } -1$$

Now we have to find out the value of $\frac{d^2y}{dx^2}$ when $x = 1$ and $x = -1$.

$$\frac{dy}{dx} = 6x^2 - 6$$

$$\frac{d^2y}{dx^2} = 6 \text{ X } 2x - 0 = 12x$$

when $x = 1$, $\frac{d^2y}{dx^2} = 12 \times 1 = 12$, a positive value.

$\therefore$ The given function reaches its minimum value at $x = 1$.

when $x = -1$, $\frac{d^2y}{dx^2} = 12(-1) = -12$, a negative value.

$\therefore$ The given function reaches its maximum value at $x = -1$.

Example 2:

Find the point at which the function $y = x^2 - 3x + 1$ reaches its maximum or minimum.

$$y = x^2 - 3x + 1$$

$$\frac{dy}{dx} = 2x - 3 + 0$$

$$= 2x - 3$$

Equating $\frac{dy}{dx}$ to zero, we get $2x - 3 = 0$. $\therefore$ $x = \frac{3}{2}$

Space for hints

$\frac{d^2y}{dx^2} = 2$, a positive value. There is no x term in $\frac{d^2y}{dx^2}$. It means, $\frac{d^2y}{dx^2}$ has a positive value for all x.

$\therefore$ The given function reaches its minimum at $x = \frac{3}{2}$

Example 3:

Find the maxima and/or minima of the function $y = x^4 - 2x^2 + 3$.

$$y = x^4 - 2x^2 + 3$$

$$\frac{dy}{dx} = 4x^3 - 4x + 0 = 4x^3 - 4x.$$

Equating $\frac{dy}{dx}$ to zero, we get

$$4x^3 - 4x = 0$$

$$4x(x^2 - 1) = 0$$

$$\therefore \ x = 0 \text{ or } x^2 - 1 = 0$$

$$x = 0 \text{ or } x^2 = 1$$

$$x = 0 \text{ or } x = +1 \text{ or } x = -1$$

$$\frac{d^2y}{dx^2} = 4 \times 3x^2 - 4 = 12x^2 - 4$$

when $x = 0$, $\frac{d^2y}{dx^2} = 0 - 4 = -4 < 0$

when $x = 1$, $\frac{d^2y}{dx^2} = 12 \times 1^2 - 4 = 12 - 4 = 8 > 0$

when $x = -1$, $\frac{d^2y}{dx^2} = 12\,(-1)^2 - 4 = 12 - 4 = 8 > 0$

$\therefore$ The given function reaches its maximum at $x = 0$ and minimum at both $x = 1$ and $x = -1$.

By substituting $x = 0$ in the given equation, we get the maximum vlaue of the function and by putting $x = \pm 1$ we get minimum value of the given function.

when $x = 0$, $y = 0^4 - 2 \times 0^2 + 3$

$= 0 - 0 + 3 = 3$

when $x = 1$, $y = 1^4 - 2 \times 1^2 + 3$

$= 1 - 2 + 3$

$= 2$

when $x = -1$, $y = (-1)^2 - 2(-1)^2 + 2$

$= 1 - 2 + 3$

$= 2$

$\therefore$ The maximum value of the given function is 3 and the minimum value is 2.

Example 4:

Find the maximum and minimum values of the function $y = 5x^3 - 3x + 1$.

$y = 5x^3 - 3x + 1$

$$\therefore \frac{dy}{dx} = 5 \times 3x^2 - 3 + 0 = 15x^2 - 3.$$

$$\frac{dy}{dx} = 0. \quad \therefore 15x^2 - 3 = 0. \qquad x^2 = \frac{3}{15} \quad = \frac{1}{5}$$

$$\therefore \quad x = \pm\sqrt{\tfrac{1}{5}}$$

$$\frac{d^2y}{dx^2} = 15 \times 2x - 0 = 30x.$$

when $x = \sqrt{\tfrac{1}{5}}$, $\quad \dfrac{d^2y}{dx^2} = 30 \times \sqrt{\tfrac{1}{5}} > 0$

when $x = -\sqrt{\tfrac{1}{5}}$, $\quad \dfrac{d^2y}{dx^2} = 30 \times \left(-\sqrt{\tfrac{1}{5}}\right) = -30 < 0$

The given function reaches its minimum at $x = \sqrt{\tfrac{1}{5}}$ and maximum at $x = -\sqrt{\tfrac{1}{5}}$

Minimum value of $y = 5\left(\sqrt{\tfrac{1}{5}}\right) - 3\left(\sqrt{\tfrac{1}{5}}\right)^3 + 1$

$$= 5 \times \frac{1}{5} \times \frac{1}{\sqrt{5}} - \frac{3}{\sqrt{5}} + 1$$

$$= \frac{1}{\sqrt{5}} - \frac{3}{\sqrt{5}} + 1 = 1 - \frac{2}{\sqrt{5}}$$

Minimum value of $y = 5\left(-\sqrt{\tfrac{1}{5}}\right)^3 - 3\left(-\sqrt{\tfrac{1}{5}}\right) + 1$

$$= -5 \times \frac{1}{5} \times \frac{1}{\sqrt{5}} + \frac{3}{\sqrt{5}} + 1$$

$$= -\frac{1}{\sqrt{5}} + \frac{3}{\sqrt{5}} + 1 = 1 + \frac{2}{\sqrt{5}}.$$

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Example 5:

Verify whether the function $y = x^2 + 2x + 2$ reaches its minimum at $x = -1$.

$$y = x^2 + 2x + 2$$

$$\therefore \frac{dy}{dx} = 2x + 2 + 0 = 2x + 2 = 2(x+1)$$

$$\frac{dy}{dx} = 0. \quad \therefore 2(x+1) = 0 \qquad x + 1 = 0 \qquad x = -1$$

$$\frac{d^2y}{dx^2} = 2 \times 1 + 0 = 2 > 0.$$

$\therefore$ The given function reaches its minimum at $x = -1$.

7. Applications of Functions and Diagrams in Economic Theory

Economics is an analytical study, concerned with the relations that exist or can be assumed to exist between quantities which are numerically measurable (e.g. prices, incomes, interest rates and so on). Therefore, economic relations are expressible by means of mathematical functions. The relations of economics and the functions which express them are usually of unspecified or unknown form. However, it is common to assume conveniently that an economic function is of linear or quadratic form (i.e) in graphical terms, the function concerned is being represented by a straight line or a U-shaped curve or inverted U-shaped curve.

Any economic problem capable of symbolic representation can be illustrated with the aid of diagrams. There is a curve corresponding to each function we use to interpret a relation between two economic variables; we have demand, cost or revenue curve as well as demand, cost or revenue functions. Diagrams displaying the properties of such curves and relating one curve to another are extremely useful.

7.1 Demand Function and Curves

Let us consider the conditions of demand on a consumers' good market and also let us take up the case where there is pure competition amongst consumers implying that each consumer acts only as a `quantitity adjuster'. In order to get a simple representation of the demand for definite good x in a market consisting of a definite group of consumers, it is usualy assumed in economic theory that

i) the number of consumers in the group

ii) the tastes of each individual consumer for all the goods in the market

iii) the income of each individual consumer, and

iv) the prices of all goods other than x itself

are all fixed and known. The amount of x each consumer will demand is considered as uniquely dependent on the price of x ruling on the market. The demand for x can change only if the market price varies. This expression of market condition can be translated into symbolic form as follows.

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Let p denote the market price of x and Q denote the quantity demanded of x. Now, the expresion Q varies with p can be written symbolically as $Q = f(p)$ and it is called the demand function for x. This demand function can also be represented graphically by means of a curve called demand curve. In $Q=f(p)$, p is the independent variable and Q, the dependent variable; according to the mathematical convention p is to be measured along the x axis and Q, along the y axis to draw the demand curve. However, since the days of Marshall, the economic convention has been otherwise and the price p is measured along the y axis and the quantity Q along the x axis.

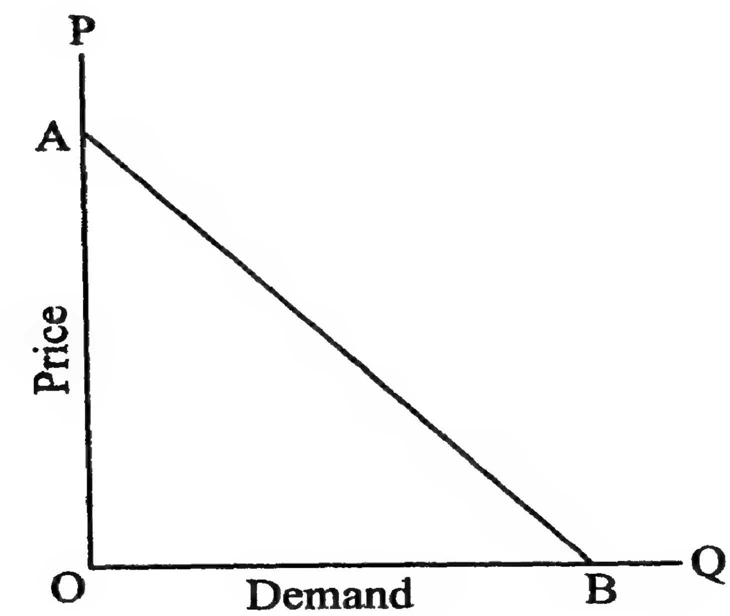
The demand relation is a completely static one and does not refer to changes in demand over time. The demand function, $Q = f(p)$ and the demand curve represent a fixed situation in which a set of alternative and hypothetical prices are given. These prices cannot be taken as actually ruling on the market at different points of time.

According to the law of demand in economic theory, larger the price of a commodity, x, the smaller the demand for x. Symbolically, this means that as p increases Q decreases. That is, the demand function is a decreasing function.

It is often convenient to represent the demand law by a definite type of function, say linear function as

$$Q = a - bp$$

and the demand curve is a straight line as follows.



In the above demand function, a and b are called parameters and the demand curve is fixed only when the values of these parameters are given definite values.

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Under given demand conditions, a and b remain as constants. When any change takes place in the demand conditions (a change in the taste of the consumer and / or income of the consumer and / or prices of other goods), values of a and b change and a shift will take place in the position of the demand curve. The nature of shift (either upward or downward) will depend upon the particular change taking place in the demand conditions and shift in the demand curve occurs only over time.

7.2 Total Revenue Function and Curve

When the demand for a commodity x is Q and the price p, the product R = Qp is called the total revenue obtainable from this demand and price. Already we saw, the demand for a good x is represented by a demand function Q = f(p). Therefore, R can be expressed as a function of p alone as

$$R = pf(p)$$

The demand function Q = f(p) can also be represented by its inverse function as p = Ø(Q). Now, R can also be represented as R = Q.Ø(Q).

Either of the two representations namely,

R = pf(p) and R = QØ(Q) is called Total Revenue Function. However, the latter representation namely, R=QØ(Q) is more convenient and commonly used. Graph of the Total Revenue Function is called the Total Revenue curve and it is drawn marking Q along the x axis and R along the y axis, the height of the curve representing the total revenue obtainable from the output indicated.

In the case of a linear demand function

$$Q = a-bp$$
$$R = p(a-bp)$$
$$= ap-bp^2$$

If we consider the inverse of the above linear demand function,

$$p = \frac{a-Q}{b}, \text{ then}$$

$$R = Q\left(\frac{a-Q}{b}\right)$$

$$= \frac{aQ}{b} - \frac{Q^2}{b}$$

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Thus, **Total Revenue Function is a quadratic function and its graph will therefore be a parabola.** Total Revenue curve is drawn with quantity marked on the x axis and revenue marked on the y axis.

When the demand function is a continuous function, total revenue function will also be a continuous function as it is derived from the demand function. However, it is not possible to indicate at this stage the `normal' form of the toal revenue function because of the decreasing nature of demand function. The form of the total revenue function depends on the concept of elasiticity of demand. R increases or decreases with increasing output according as the demand is elastic or inelastic.

7.3 Cost Function and Curves

As far as the cost function is concerned `time' is an important factor to be taken into consideration. That is, we have to consider whether the production activities taking place in the short run or in the longrun are taken into account. Because, in the short run, some of the factors are employed in fixed amounts irrespective of the level of output of the firm and the expenditure on these factors is known and fixed; the remaining factors vary in quantity according to the level of output and the conditions of their supply are assumed to be known (for example, the factors may be obtainable at fixed market prices). On the other hand, in the longrun, all the factors are variable factors, that is, all the factors vary in quantities according to the level of output.

7.3.1 Cost function in the shortrun

Total cost of the production is usually denoted by the symbol p and it consists of two components namely, Total Fixed Cost (TFC) and Total Variable Cost (TVC). That is,

$$\pi = TFC + TVC$$

$$TFC = K, \text{ a constant}$$

$$TVC = f(Q)$$

$$\therefore \pi = K + f(Q)$$

$$= \varnothing(Q), \text{ a function of } Q$$

Cost curves are drawn with quantity of output measured along the x axis and cost along the y axis. Total fixed cost curve is a horizontal straighline as follows.

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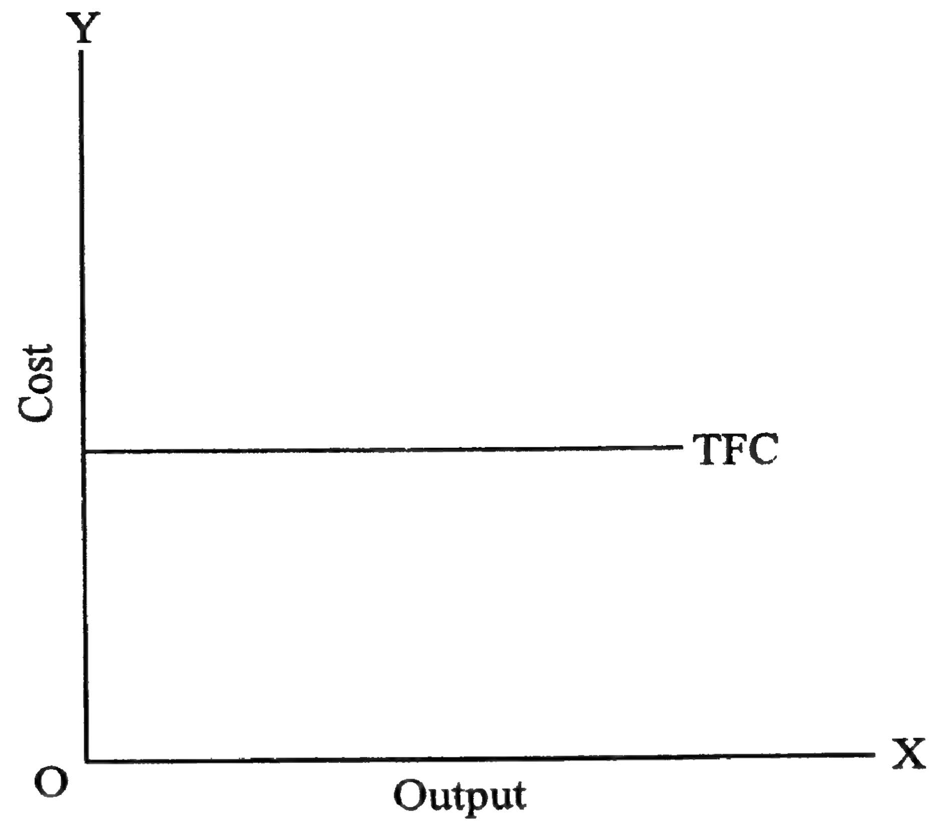
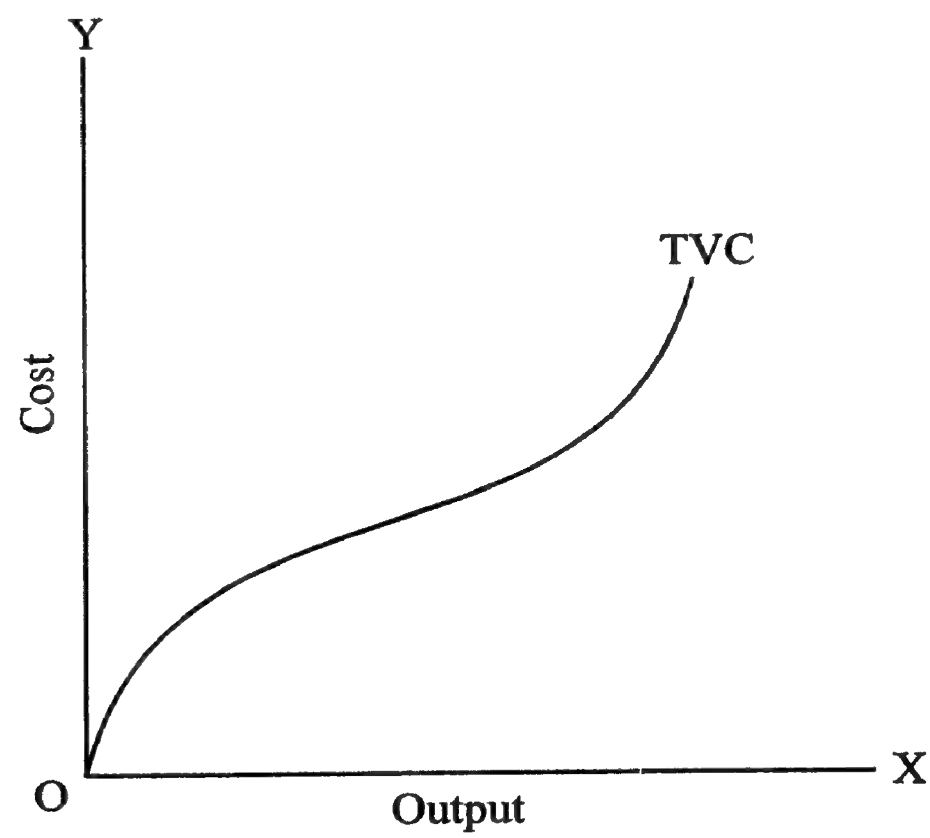
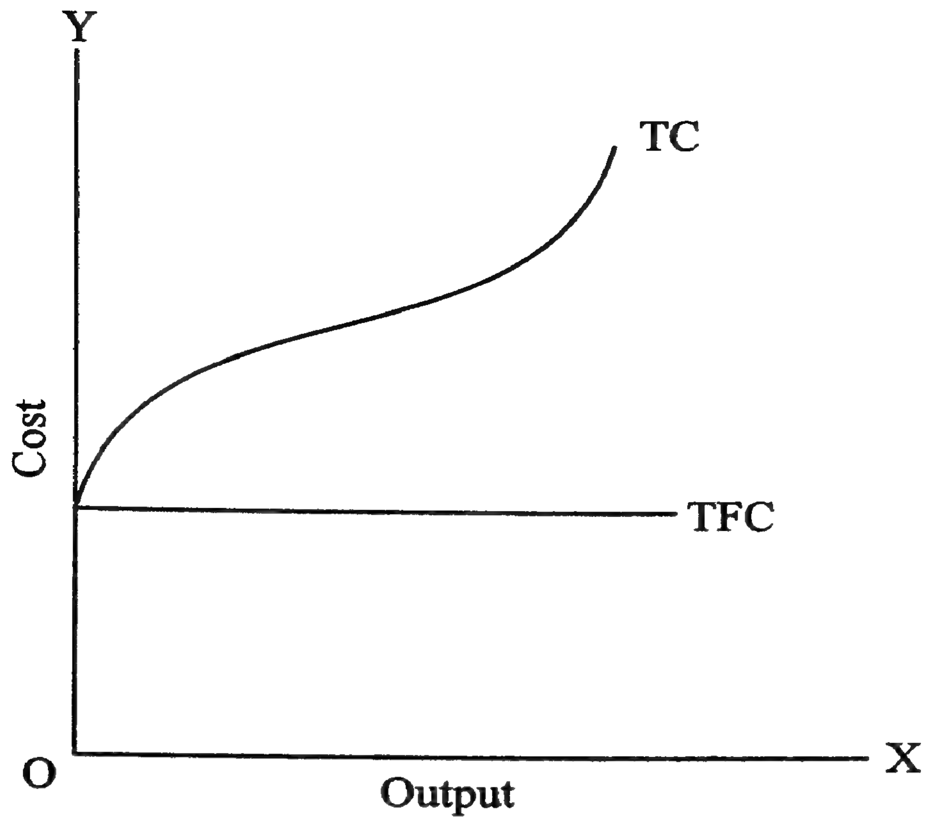


Total variable cost increases with increasing level of output and is equal to zero when no output is produced. Therefore, TVC curve is a curve going upwards from left to right starting from the origin as follows.



Total cost curve is the sum of TFC and TVC curves and therefore, it also goes upwards starting not from the origin but from the starting point of TFC curve as follows.



7.3.2 Cost function in the long run

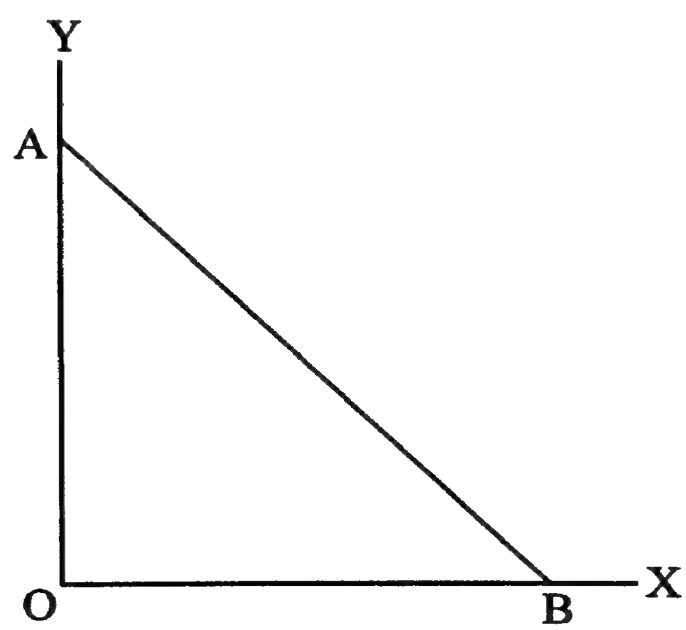
Long run total cost (LTC) function is given as

$$\pi = f(Q)$$

where π is the total cost and Q is level of output. LTC curve also goes upwards starting from the origin.

7.4 Concept of slope and derivative

Already saw that 'slope' is a mathematical term used to represent the degree of steepness (or flatness) of a curve. When the given curve is a straight line, slope is a constant and its value is obtained as follows.



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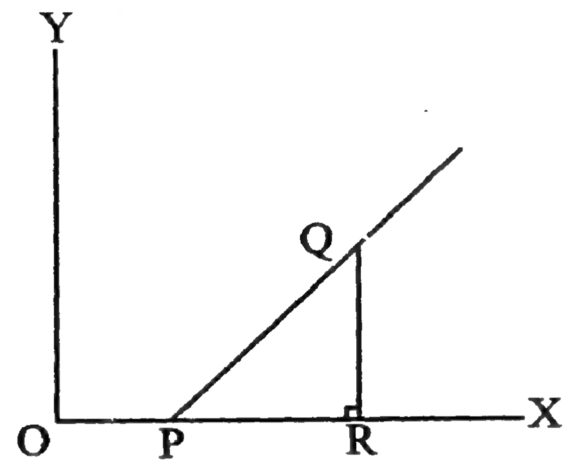
AB is a straight line going downwards meeting y axis at A and x axis at B. Here,

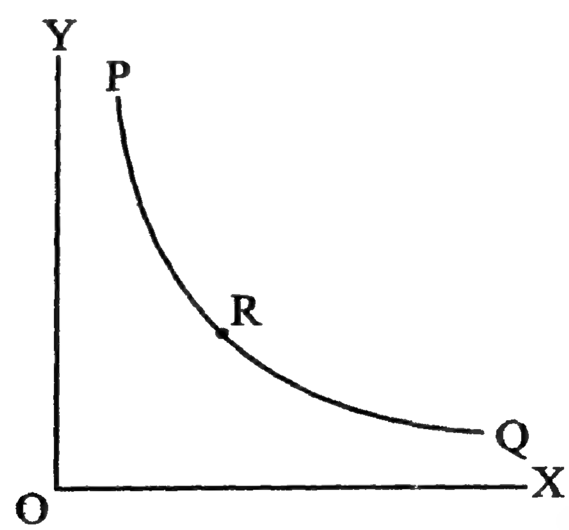
$$\text{Slope of AB} = \frac{OA}{OB}$$

Suppose the given straight line goes upwards meeting x axis at a point P and Q be any point lying on it as shown in the following diagram. Let a vertical line be drawn through Q to meet the x axis at R.

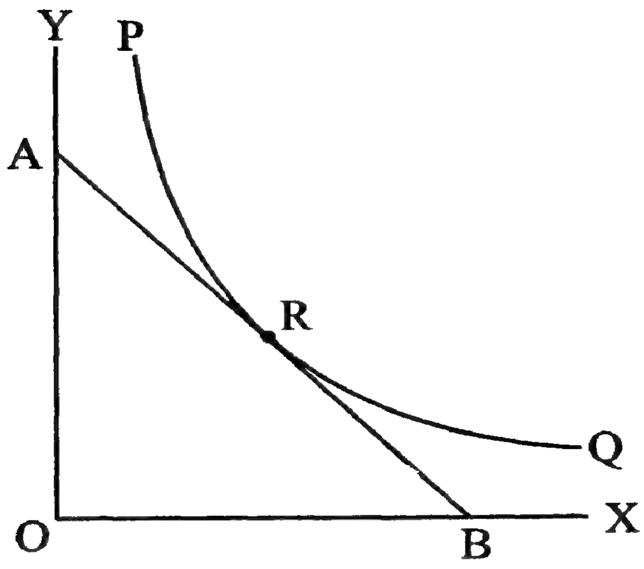


$$\text{Now, slope of the line PQ} = \frac{QR}{PR}$$

Instead of a straight line, if a curve is given, slope of the curve varies from point to point. Therefore, it is meaningful if only we speak about the slope of the curve with reference to a particular point on the curve. Consider the following diagram.



In the above diagram, PQ is a curve and R is a point lying on the curve. To compute the slope of PQ at point R, first of all a tangent to the curve PQ at R (i.e. a straight line just touching PQ at R) is drawn. Suppose the tangent meets the Y axis at A and X axis at B as follows.



Now, slope of PQ at R = slope of the tangent AB at R. $= \dfrac{OA}{OB}$

Whether a straight line or a curve represents the relationship between two given variables, the slope at a given point is obtained as the value of the derivative of the function at the given point. That is, slope $= \dfrac{dy}{dx}$.

7.5 Concept of elasticity and derivative

Elasticity of a function y = f(x) is defined as percentage change in y with respect to one percent change in x, when the given function is a discrete one, elasticity is given symbolically as follows.

$$\text{Elasticity of y w.r.t. } x = \frac{\dfrac{\Delta y}{y}\times 100}{\dfrac{\Delta x}{x}\times 100}$$

$$= \frac{x}{y}\cdot\frac{\Delta y}{\Delta x}$$

Here Dx represents a given change in x and Dy, the corresponding change in y. Elasticity of a function given above may also be rewritten as,

$$\text{Elasticity of y w.r.t } x = \frac{\left(\dfrac{\Delta y}{\Delta x}\right)}{\left(\dfrac{y}{x}\right)}$$

The numerator is rate of change of y w.r.t x and the denominator is average value of y per unit value of x. That is,

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Elasticity of a function [$y = f(x)$] is $= \dfrac{\text{Rate of change of y w.r.t. x}}{\text{Average value of y per unit value of x}}$

In the case of continuous function, rate of change of y w.r.t. x is given by the derivative $\dfrac{dy}{dx}$.

$$\therefore \text{Elasticity} = \frac{\left(\frac{dy}{dx}\right)}{\left(\frac{y}{x}\right)}$$

7.6 Applications of the concepts of slope and derivatives in Economics

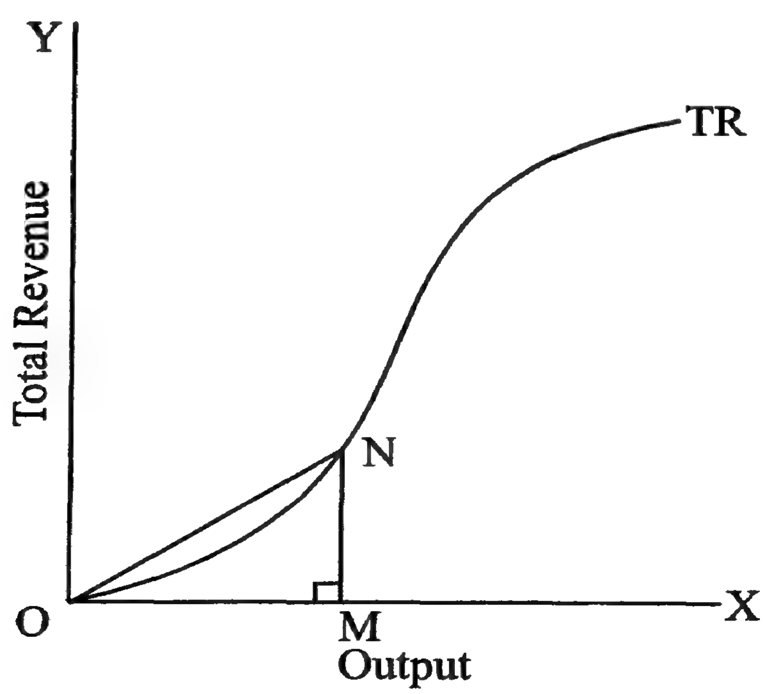
Earlier we saw Total Revenue and Total Cost function and their curves. In Economics, we come across two more related concepts namely, `Average' and `Marginal' with reference to these two `Total' concepts. That is, in respect of revenue, average revenue and marginal revenue, and in respect of cost, average cost and marginal cost. How the average and marginal functions / curves are related to the respective total are discussed below.

7.6.1 Average Revenue function

Average revenue is defined as the revenue received on an average by the firm from the sale of one unit of output. Therefore if $R = Q. \varnothing(Q)$ is the total revenue function,

$$\text{Average revenue} = \frac{R}{Q} = \varnothing(Q).$$

From this, it is clear that average revenue function is nothing but the demand function and average revenue is the same as the price of the commodity.



Check your Progress

3. Define Elasticity of a function.

Graphically speaking, given the total revenue curve, average revenue at a given level of output is obtained as the slope of the straight line joining the point on the TR curve corresponding to the given output and the origin. This can be easily understood with the help of the above diagram.

In the above diagram, TR is the total revenue curve. Let OM be the given level of output and N, the corresponding point on TP curve. Now, the value of TR in respect of the level of output OM is equal to NM

$$\text{AR at OM output} = \frac{TR}{Q} = \frac{NM}{OM}$$

$$= \text{Slope of ON}$$

$$= \text{Slope of straight line joining origin and the point on TR.}$$

7.6.2 Average cost function :

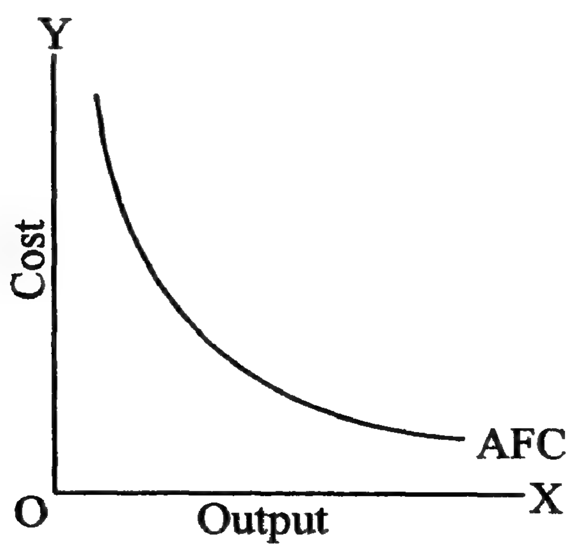
Average cost (AC) is defined as cost incurred per unit of output by the firm.

That is, $$AC = \frac{\pi}{Q}$$

In the shortrun, π = TFC + TVC. Therefore, short run average cost (SAC) is given by

$$SAC = \frac{TFC+TVC}{Q} = \frac{TFC}{Q}+\frac{TVC}{Q} = AFC+AVC$$

As TFC is constant for all levels of output, AFC = $\frac{TFC}{Q}$ is a decreasing function and the corresponding curve is a special curve called 'rectangular hyperbola' which is illustrated below.



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Check your Progress

4. Define Average Revenue function.

5. What is Average Cost function?

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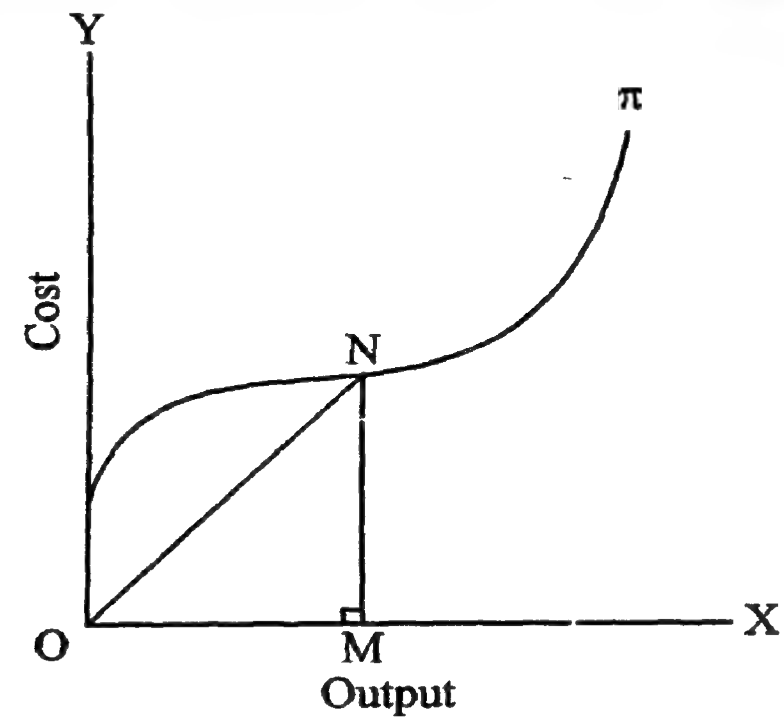
The special property of this curve is that the two ends of the curve go closer and closer to the x and y axes but they never meet the two axes. In mathematical terms, X and Y are **'asymptotes'** to the AFC curve.

As far as AVC is concerned, according to economic theory, it is a decreasing function upto a certain level of output and after that, it is an increasing function. In short, AVC = $\frac{TVC}{Q}$ is represented by a U shaped curve. Mathematically speaking, AVC curve is a **'parabola'** and hence, AVC function is represented by a quadratic function.

As SAC is the sum total of AFC and AVC, it will also be a U shaped curve (that is, parabola) and SAC will reach its minimm after AVC reaches its minimum.

Just like in the case of average revenue, average cost can also be computed from the graph of total cost curve. Suppose OM is the given level of output and π, the total cost curve in the following diagram.

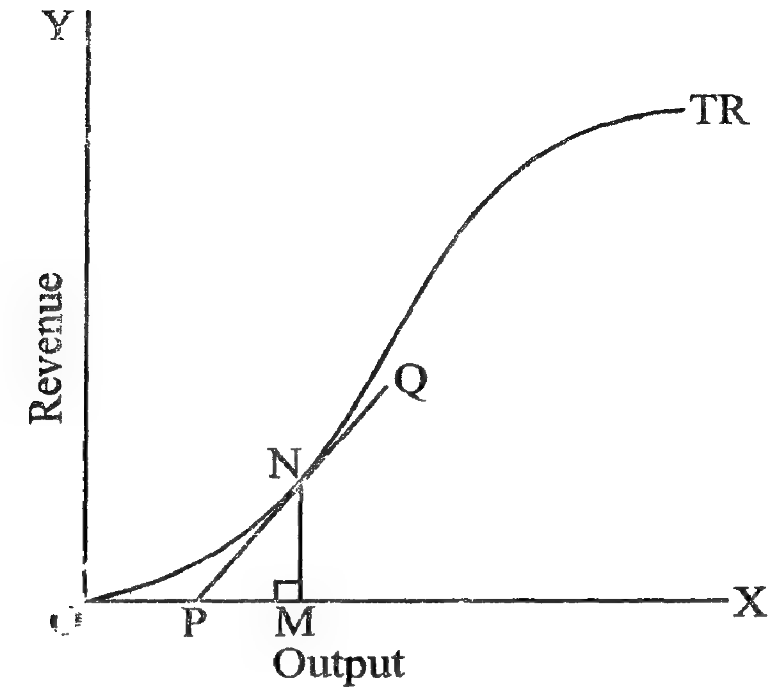


AC at OM output = $\frac{NM}{OM}$ = slope of ON

7.6.2 Marginal Revenue function

In economic theory, marginal revenue is defined as the **addition made to total revenue** when an additional unit of output is sold in the market. In mathematical terms, marginal revenue represents the rate of change of total revenue with respect to changes in total output. In terms of derivative, marginal revenue (MR) is equal to the derivative of the total revenue function. Symbolically, if R = F(Q) is the total revenue function then MR = $\frac{dR}{dQ}$.

It was stated earlier that derivative of a function represents the slope of the curve. It is derived from this fact that MR may be obtained as the slope of the TR curve. When TR curve is not a straight line its slope varies from point to point and the slope at a given point is computed as the slope of the tangent drawn at that point. It is illustrated in the following diagram.



N is the given point on the TR curve corresponding to the output OM. At N, to get the value of MR, a tangent PQ is drawn to TR curve at N. It meets x axis at P. Now,

$$\text{MR at N} = \text{Slope of tangent PQ at N}$$

$$= \frac{NM}{PM}$$

7.6.4 Marginal cost function

Just like marginal revenue, marginal cost (MC) is **addition made to total cost** when an additional unit of output is produced. That is, MC represents the **rate of change** in the total cost when output expands and hence, MC is equal to the derivative of total cost function. Symbolically,

$$MC = \frac{d\pi}{dQ}$$

Given the total cost curve, MC with respect to a given level of output is obtained as the slope of the TC curve corresponding to the given level of output. That is,

MC at output OM = Slope of tangent to TC curve at output OM.

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Check your Progress

6. What is Marginal Revenue function?

7. What is Marginal Cost function?

Space for hints

8. Elasticity concepts in Economics

8.1 Elasticity of Demand

The concept of elasticity of demand is generally associated with the name of Alfred Marshall, though it was evolved long before by economists like Cournot and Mill.

In the words of Marshall, "The elasticity" (or responsiveness) of demand in a market is great or small according as the amount demanded increases much or little for a given fall in price. Elasticity of demand is therefore a technical term used by the economists to describe the degree of responsiveness of the demand for a commodity to a change in price. If Ed stands for the price-elasticity of demand, the formula to find out the numerical value of price-elasticity of demand is given as follows :

$$Ed = \frac{\text{Proportionate change in quantity demanded}}{\text{Proportionate change in price}}$$

$$= \frac{\text{Change in Quantity } (\Delta Q)}{\text{Initial Quantity } (Q)} \div \frac{\text{Change in Price } (\Delta P)}{\text{Initial Price } (P)}$$

$$= \frac{-\Delta Q}{Q} \div \frac{\Delta P}{P} \text{ (since Q and P are inversely related)}$$

$$= -\frac{\Delta Q}{\Delta P} \times \frac{P}{Q}$$

When the demand function is known and given to us,

Ed formula is given as $Ed = \frac{-dQ}{dP} \times \frac{P}{Q}$

If the proportionate change in quantity demanded and price are known the numerical value of Ed can be calculated.

On the basis of the numerical value, elasticity of demand for a commodity has been broadly classified into two as Elastic demand and Inelastic demand. If a change in the price of commodity causes a more than proportionate change in quantity demanded, that demand is said to be elastic; in other words, when the numerical value of elasticity of demand is greater than one, demand is elastic and the demand curve is relatively flatly sloped.

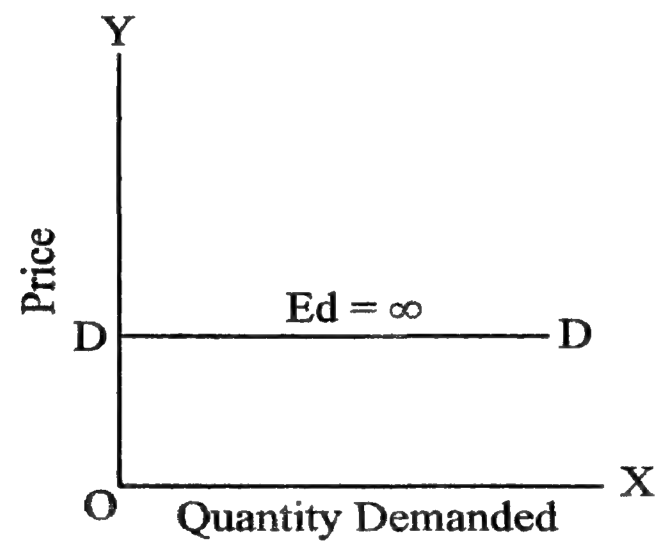
On the other hand, if a change in the price of a commodity causes a less than proprotionate change in quantity demanded, then demand is said to be inelastic; in other words, when the numerical value of elasticity of demand is less than one, demand is inelastic. When the demand is inelastic the demand curve will generally be relatively steeply sloped.

If a given proportionate change in price is accompanied by an exactly equal proportionate change in quantity demanded, elasticity of demand becomes unity (equal to one). In the case of unit elasticity of demand, the demand curve assumes the shape of a rectangular hyperbola and the demand equation takes the form, QP = k, a constant.

Apart from the above three different cases of elasticity of demand there are two other limiting or extreme cases, namely, infinitely or perfectly elastic demand and zero elastic demand or perfectly inelastic demand.

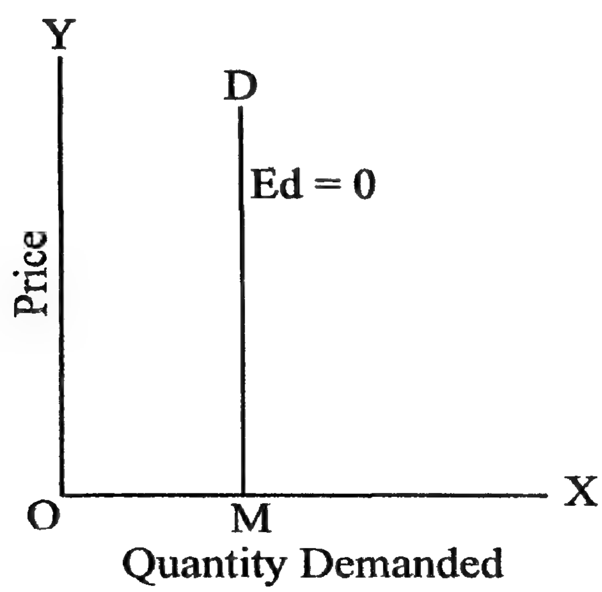
Perfectly elastic demand (Ed = ∞)

In this case, a very small change in the price of the commodity leads to infinite change in quantity demanded. When demand is perfectly elastic, the demand curve will be a horizontal straight line parallel to X axis as shown below.



Perfectly inelastic demand (Ed = 0)

If a change in price has no effect on the quantity demanded it is the case of zero elastic demand or perfectly inelastic demand. When demand is perfectly inelastic, the demand curve will be a vertical straight line parallel to Y axis, as shown below :



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8. What do you mean by Elasticity of demand?

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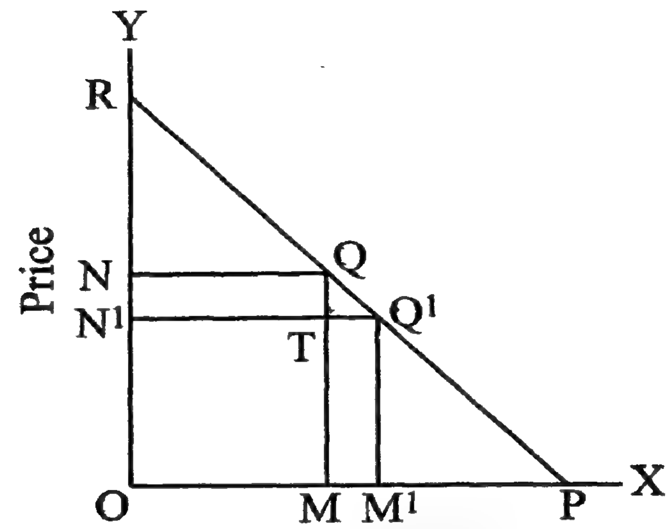
8.2 Measurement of Price Elasticity of Demand - Graphical Method

Case (i) : Straight line demand curve

This method of measuring price elasticity of demand has also been suggested by Alfred Marshall. According to this method, we take a straight line demand curve joining the two axes and measure the elaticity between two points Q and Q1 which are assumed to be very close to each other.



In the above figure RP is the straight line demand curve which connects both the axes. At the original price ON the quantity demanded is OM. Then price changes to ON^1 and the new quantity demanded is OM^1.

Then, the price elasticity of demand,

$$Ed = \frac{\Delta Q}{\Delta P} \times \frac{P}{Q}$$

$$= \frac{MM^1}{NN^1} \times \frac{ON}{OM} \qquad \text{------ (1)}$$

In the above diagram, the small triangle QTQ^1 is similar to the bigger triangle QMP. Therefore, the ratio to corresponding sides of these triangles is the same.

i.e., $$\frac{QT}{QM} = \frac{TQ^1}{MP}$$

But, $QT = NN^1$, $QM = ON$ and $TQ^1 = MM^1$

$$\therefore \quad \frac{NN^1}{ON} = \frac{MM^1}{MP}$$

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$$\therefore \quad \frac{MM^1}{NN^1} = \frac{MP}{ON}$$

Putting this in (1), we get

$$Ed = \frac{MP}{ON} \times \frac{ON}{OM} = \frac{MP}{OM}$$

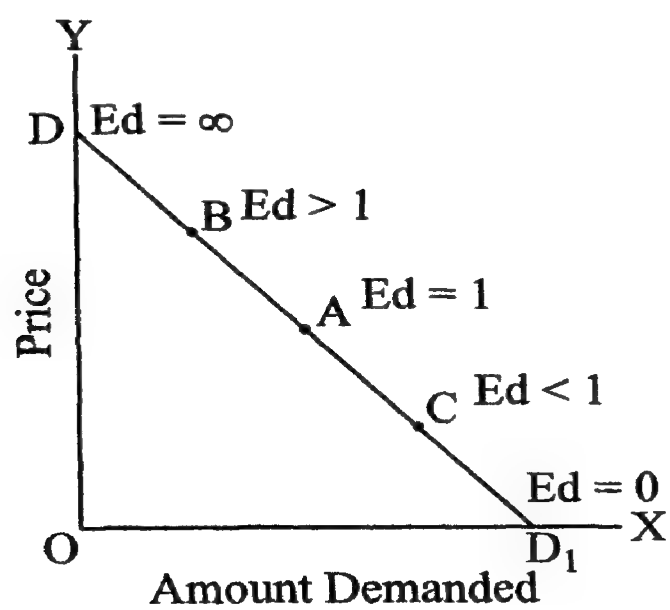
In triangle OPR, MQ is parallel to OR.

$$\therefore \quad \frac{MP}{OM} = \frac{PQ}{QR}$$

Hence $\quad Ed = \dfrac{PQ}{QR}$

If the distance between Q and Q^1 is reduced such that Q^1 tends to coincide with Q, Ed represents elasticity of demand at Q. We can find out even numerical values of elasticity of demand at different points of the striaght line demand curve with the help of this formula.

In figure below DD is the straight-line demand curve. A is the mid-point of DD_1 curve. Applying the above formula, price elasticity of demand at point A $= \dfrac{AD^1}{AD} = 1$ because $AD = AD_1$.



Let us take another point B on the DD_1 curve. The elasticity of demand at B $= \dfrac{BD^1}{BD}$ which is greater than 1 because $BD_1 > BD$.

Space for hints

On the other hand, at any point lower than mid-point A, elasticity will be less than 1. For example, elasticity at point C will be equal to $\frac{CD^1}{CD}$ which is less than unity because $CD_1 < CD$.

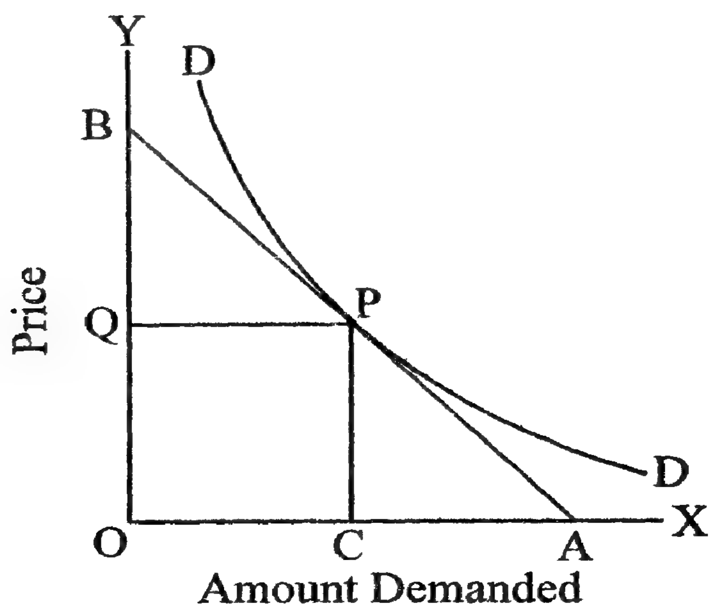
Elasticity at the point where the demand curve meets the Y axis, i.e., at point D, the elasticity of demand $\frac{DD^1}{DD} = \frac{DD^1}{zero}$ i.e., infinity. Lastly, elasticity at the point where the demand curve meets the X axis i.e., at point D^1 will be equal to $\frac{D^1D^1}{DD^1} = \frac{zero}{DD^1}$ = zero.

Thus we can conclude that at mid-point on a straight-line demand curve, elasticity will be equal to unity. At the higher points on the same demand curve, to the left of the mid-point elasticity will be greater than unity; at lower points, points on the demand curve to the right of the mid-point, elasticity will be less than unity. At the point where the demand curve meets Y axis, elasticity is infinite; and at the point where it meets the X axis, elasticity is Zero.

Case (ii) : Non-linear demand curve

If the demand curve is not a straigh line, then elasticity of demand at a point P on the demand curve is calculated as follows.

Suppose DD is the demand curve and P is any point on DD. Draw a tangent BA to the demand curve at point P as shown in the following diagram.



Space for hints

Elasticity of demand Ed $= \frac{\Delta Q}{\Delta P} \times \frac{P}{Q}$. Here, the term $\frac{\Delta Q}{\Delta P}$ is actually reciprocal of the slope of the Demand curve at point P. The slope of the demand curve at P = The slope of the tangent BA in the figure above is equal to $\frac{PC}{CA}$.

$$\therefore \quad \frac{\Delta Q}{\Delta P} = \frac{1}{\text{Slope of the tangent BA}} = \frac{CA}{PC}$$

In above figure PC is the price level and OC is the quantity demanded of X.

$$\therefore \quad \frac{P}{Q} = \frac{PC}{OC}$$

Therefore, $$Ed = \frac{PC}{OC} \times \frac{CA}{PC} = \frac{CA}{OC}$$

In triangle BOA, PC is parallel to OB.

$$\therefore \quad \frac{CA}{OC} = \frac{PA}{PB}$$

$$\therefore \quad Ed = \frac{PA}{PB}$$

8.3 Income Elasticity of Demand

Income effect, a concept introduced by Prof.J.R. Hicks explains the functional relationship between income of the consumer and amount demanded of a commodity. Under normal circumstances and in the case of the ordinary and normal goods, with every increase in the income of the consumer, amount demanded of a commodity will be rising; with every decline in the income of the consumer amount demanded of a commodity will also be declining. Thus other things remaining the same, there is a direct or positive relationship between income and amount demanded of a commodity.

Corresponding to the Hicksian concept of income effect, there is a concept of income elasticity of demand. Income elasticity of demand is a measure of the responsiveness of demand to a given change in the income of the consumer, other thigns remaining constant. It compares the rate of change in the purchase of a commodity say X with the rate of change in the

Space for hints

income of the consumer. Hence, it is a ratio of proportionate change in the purchase of commodity X to a proportionate change in income. The formula generally used to measure the income elasticity of demand for any good X (denoted by E_I) is as follows :

$$E_I = \frac{\text{Proportionate change in the purchase of commodity X}}{\text{Proportionate change in the consumer's income}}$$

$$= \frac{\frac{\Delta x}{x}}{\frac{\Delta y}{y}} = \frac{\Delta x}{\Delta y} \times \frac{y}{x}$$

We can distinguish between three varieties of income elasticity of demand namely (1) $E_I = 1$ (2) $E_I > 1$ and (3) $E_I < 1$. If the proportion of the consumer's income spent on X commodity is exactly the same before and after a change in income, income elasticity is said to be equal to 1. If the consumer spends an increasing proportion of his money income on X commodity as he becomes richer income elasticity will be greater than 1. If the proportion of income spent on the commodity X falls with a rise in the income of the consumer, income elasticity will be less than 1.

8.4 Cross Elasticity of Demand

The idea of elasticity of demand can also be applied in a situation where two commodities are related to each other in some way. The concept of cross elasticity of demand is useful in handling intercommodity relations. This serves as a measure of demand with interdependence between goods.

Cross elasticity of demand measures the responsiveness of amount demanded of one commodity to a given change in price of another commodity. It indicates the extent to which a proportionate change in the price of commodity X causes a proportionate change in the purchase of commodity Y. The following is the formula to find out the numerical value of cross elasticity of demand.

Check your Progress

9. Define Income Elasticity of demand.

10. What is Cross Elasticity of demand?

$$\text{Cross Elasticity of demand (Ec)} = \frac{\text{Proportionate change in the purchase of Y}}{\text{Proportionate change in the price of X}}$$

The concept of cross elasticity of demand applies to competitive goods or substitutes and complementary goods. Generally speaking, in the caseof substitutes say, X and Y, a fall in the price of commodity X, will lead to a fall in the amount demanded of Y and Vice-versa.

But in the case of complementary goods say, X and Z, to a given decline in the price of commodity X there will be a rise in the quantity demanded of commodity Z and vice versa.

Space for hints

8.5 Elasticity of substitution between commodities

8.5.1 Computational Procedure

The concept of elasticity of substitution plays an important role in modern economic theory. Substitution elasticity is the corresponding concept of substitution effect of indifference curve analysis. Substitution elasticity may be defined as the extent to which one commodity can be substituted for another as a consequence of a given change in their price ratio if the consumer is to enjoy the same amount of satisfaction. The ratio in which the two commodities are bought or combined in naturally changes when the process of substitution is taking place. The elasticity of substitution is thus the ratio of the proportionate change in the combination which the two commodities are bought to a given proportionate change in their price ratio.

We illustrate this concept with the help of the following imaginary example. Let us suppose that two substitutes. X and Y, are being sold at Rs.5 and Rs.10 per Kg. respectively. At these prices let us suppoes that a consumer buys 3 Kilograms of X and 2 Kilograms of Y. Therefore, the ratio of X, Y purchased by the consumer is $\frac{X}{Y}=\frac{3}{2}$ while their price ratio is $\frac{Px}{Py}=\frac{5}{10}$ or $\frac{1}{2}$. Now let us further suppose that the price of Y rises to Rs.15 per kg., while the price of X remains at Rs.5 per kg. The consumer would therefore buy less of Y and purchase more of X. He will re-allocate his expenditure by substituting X for Y. How far shall he go in the direction depends upon the extent to which Y can be substituted by X. Let us suppose that he buys now 5 kilograms of X and 1 Kilogram of Y. The new X, Y ratio is $\frac{5}{1}$ while their price ratio $\frac{5}{15}=\frac{1}{3}$. Elasticity of substitution is,

$$= \frac{\text{Proportionate change in the ratio of quantities of goods}}{\text{Proportionate change in the price ratio}}$$

$$= \left[\frac{\Delta\left(\frac{x}{y}\right)}{\frac{x}{y}}\right] \div \left[\frac{\Delta\left(\frac{Px}{Py}\right)}{\frac{Px}{Py}}\right]$$

Space for hints

$$= \left[\frac{\frac{5}{1}-\frac{3}{2}}{\frac{3}{2}}\right] \div \frac{\left[\frac{1}{3}-\frac{1}{2}\right]}{\frac{1}{2}}$$

$$= \left[\frac{7}{3}\right] \div \left[-\frac{1}{3}\right]$$

$$= -7$$

8.5.2 Types

(A) Perfect Elastic :

The substitution elasticity will be **infinite** if two commodities are **perfect substitutes** for each other. In that case, a fall in the price of one commodity assuming the price of the other commodity to be constant will lead the consumer to substitute the former completely for the latter and the indifference curve will be a straight line. But in actual life we rarely come across two commodities which are perfect substitutes.

(B) Perfect Inelastic :

If two commodities are not substitutes for each other, then elasticity of substitution will be zero. In such a caes the rate of substitution between them will be zero.

(C) High or Low Elasticity :

Between the above two exterme limits of infinite and zero elasticity, we can have high or low substitution elasticities. If the rate of substitution falls gradually, the elasticity of substitution will be low. On the contrary, if the rate of substitution falls rapidly, the elaticity of substitution will be high.

8.6 Elasticity of Input substitution

There is another important elasticity concept associated with the production function that is useful in decision making by the firm, the elasticity of input substitution. It is defined as the ratio of the percentage change in the input ratio and the percentage change in the marginal rate of technical substitution, when output is held constant and it is denoted by s. Symbolically,

$$\sigma = \frac{\dfrac{\Delta\left(\frac{C}{L}\right)}{\left(\frac{C}{L}\right)}}{\dfrac{\Delta(\text{MRTS})}{(\text{MRTS})}}$$

Space for hints

Like the elasticity of demand, elasticity of substitution also has three ranges : An inelastic range, an elastic range and a unit elasticity range. Elasticity of input substitution measures the ease with which one input may be substituted for another along a particular isoquant.

It helps in determining how easy it would be to change the technique of production. If $\sigma = 0$, it indicates that for technological purposes, the firm is unable to change the mix in which it uses inputs. Even if the relative prices of inputs change, the production technologies will not allow the firm to take advantage of the change. Thus, the value of σ roughly indicates which industries will be able to hold their costs down in the face of rising relative input prices by changing techniques of production and which industries will incur increases in production costs which must ultimately reduce profits or increase output prices. If the value of σ is greater, better is the possibility for the industry to protect consumers against product price increases.

Value of σ is related to the shape of the isoquant. If σ is closer to zero, the isoquant resembles more to right angles. As σ becomes very large, the isoquants look more like straight lines.

Example

If $p = 12 - \frac{1}{3}q$, find out the TR and MR functions. Also find out the elasticity of demand when p = 10 and q = 6.

Answer

Given demand function is

$$p = 12 - \frac{1}{3}q$$

TR function is

$$R = pq$$

$$= \left(12 - \frac{1}{3}q\right)q$$

$$= 12q - \frac{1}{3}q^2$$

MR function is

$$MR = \frac{dR}{dq}$$

$$= \frac{d}{dq}\left(12q - \frac{1}{3}q^2\right)$$

Space for hints

$$= 12-\frac{2}{3}q$$

Elasticity of demand, $E_d = -\frac{dq}{dp}\cdot\frac{p}{q}$

Differentiating both sides of demand function w.r.t. q we get

$$\frac{dp}{dq} = 0-\frac{1}{3} = -\frac{1}{3}$$

$$\therefore \frac{dq}{dp} = -3$$

$$\therefore E_d = -(-3)\frac{p}{q} = 3\frac{p}{q}$$

when p = 10 and q = 6, $E_d = 3\times\frac{10}{6} = 5.$

Example

If $p = a-bq^2$, at what point MR = 0

Answer

Given demand function is

$$p = a-bq^2$$

$\therefore$ TR function is, $R = pq$

$$= (a-bq^2)q$$

$$= aq-bq^3$$

$$MR = \frac{dR}{dq}$$

$$= \frac{d}{dq}(aq-bq^3) = a-3bq^2$$

when MR = 0, $a-3bq^2 = 0$

$$a = 3bq^2$$

$$\therefore q^2 = \frac{a}{3b}$$

$$q = \pm\sqrt{\frac{a}{3b}}$$

But, as q denotes quantity of output, it cannot be negative. $\therefore$ MR = 0 at $q = \sqrt{\frac{a}{3b}}$.

Example

The demand function at point $x = 0$ and $p = 3$ is given by $x = 18-2p^2$. If the price decreases by 5% determine the relative increase in demand and hence an approximation to the elasticity of demand. (Note : In this problem, instead of q, x is given).

Space for hints

Answer :

Initial price $p = 3$ be denoted by p_1 and the price after 5% decrease be denoted by p_2. That is,

$$\frac{p_2 - p_1}{p_1} \times 100 = -5$$

$$\frac{p_2 - 3}{3} = \frac{-5}{100} = \frac{-1}{20}$$

$$p_2 - 3 = \frac{-3}{20}$$

$$\therefore p_2 = \frac{-3}{20} + 3$$

$$= \frac{-3+60}{20}$$

$$= \frac{57}{20}$$

$$= 2.85$$

Given demand function is $x = 18-2p^2$

When $p = p_2 = 2.85$,

$$x = 18-2(2.85)^2$$

$$= 18-16.245$$

$$= 1.755$$

$\therefore$ When there is a 5% decrease in price from $p = 3$, demand increases from $x = 0$ to $x = 1.755$

Elasticity of demand is given by the formula,

$$Ed = \frac{-dx}{dp} \times \frac{p}{x}$$

Space for hints

Differentiating the given demand function,

$$x = 18 - 2p^2, \text{ we get}$$

$$\frac{dx}{dp} = -4p$$

$$\therefore E_d = -(-4p)\frac{p}{x}$$

$$= \frac{4p^2}{x}$$

$$\text{When } p = 2.85, \quad x = 1.755$$

$$\therefore E_d = \frac{4(2.85)^2}{1.755}$$

$$= \frac{4 \times 8.1225}{1.755}$$

$$= 18 \text{ (approx.)}$$

9. Answers to Check Your Progress Questions :

1. Refer 1
2. Refer 2
3. Refer 7.5
4. Refer 7.6.1
5. Refer 7.6.2
6. Refer 7.6.3
7. Refer 7.6.4
8. Refer 8.1
9. Refer 8.3
10. Refer 8.4

10. Model questions for guidance :

10 Marks Questions (One Page Answer)

1. Find $\frac{dy}{dx}$

a) $y = 3x^4 - x^2 + 1$

b) $y = \frac{4x^2}{x^3+5}$

2. Find $\frac{dy}{dx}$

a) $y = 16x^2 - 15x + 8$

b) $y = \dfrac{4x^3 - 7x^2 - 8}{4x + 3}$

Space for hints

3. Find $\frac{dy}{dx}$

a) $y = \log(4x^2)$

b) $y = \dfrac{x-2}{x+7}$

4. a) If $TC = 9Q^2 - 10Q + 50$ and $Q = 20$, find MC.

b) If $TR = 6Q^3 - Q^2 + 5Q$, find AR and MR.

5. What is the differential coefficient of a function. Explain its applications in Economics.

6. Explain the derivative of a function expressed graphically.

7. Explain the conditons under which a function reaches its (a) maximum and (b) minimum.

8. Explain the concept of slope and its applications in Economics.

9. Graphically explain the relationship between TR, AR and MR functions.

10. Given $Q = ax^2+by^2$. Find whether it is a homogeneous function. If so, what is the degree of homogeneity?

11. If $Q = AK^{\alpha}L^{1-\alpha}$, name the type of this function and give reason for your answer.

20 Marks Questions (Three Page Answer)

1. Find $\frac{dy}{dx}$

a) $y = (1-\sqrt{x})(1+\sqrt{x})$

b) $y = \log\sqrt{1-x^2}$

c) $y = e^{5x^2+2x+2}$

Space for hints

2. Find

$$y = \frac{1}{3}x^3 - 2x^2 + 4x + 1$$

3. Find

$$\pi = q^3 + 48q^2 + 180q - 800$$

4. Find

a) $D = p^2 - 5p + 32$ if $p = 8$

b) $C = -5Q^3 + 2Q^2 + 3Q$ IF $Q = 2$.

Define elasticity of a function. Explain the application of elasticity concept in Economics.

About the Examination

As we have already mentioned in the Introduction, Quantitative Techniques is one among the three subjects under Part III on which you have to write examination at the end of your second year course. The examination duration will be 3 hours and the maximum marks is 100. The question paper will consist of two sections namely, section A and section B. In section A, you will be given 8 questions and you have to answer any 4 questions and each answer will carry a maximum of 10 marks. That is, maximum marks for answers to questions in section A is 4 x 10 = 40. In section B, you have to answer any 3 questions out of 6 questions given and each will carry 20 marks. That is, 3 x 20 = 60. Therefore, the **maximum marks** for the subject is 40 + 60 = 100. The **passing minimum** for the subject is **35 marks**.

Eligibility for the Degree

1. A candidate will be eligible for the B.A. Degree by completing 3 years and passing all the prescribed examinations.

2. A candidate shall be declared to have passed the course if he/she scored a minimum of 35% marks in each subject.

3. A candidate shall have been declared to have passed in

(a) I Class if he/she obtains an average of 60% and above.

(b) II Class, if he/she obtains an average of 50% and above but less than 60%.

(c) III Class, if he/she obtains an average of 35% and above but less than 50%.

QUANTITATIVE TECHNIQUES

MAY 2005

Time: Three hours | Maximum: 100 marks

SECTION A - (4 x 10 = 40 marks)
Answer any FOUR questions.
All questions carry equal marks.

1. (a) Define a function.

   (b) Solve

   $$2x - 3y = 1$$
   $$5x + 4y = 14$$

2. Find $\frac{dy}{dx}$

   (a) $y = 3x^4 - x^2 + 1$

   (b) $y = \frac{4x^2}{x^3 + 5}$

3. Solve by Cramer's rule.

   $$5x - 2y + 3z = 16$$
   $$2x + 3y - 5z = 2$$
   $$4x - 5y + 6z = 7$$

4. Explain random sampling and any two methods followed in it.

5. Write about the merits and demerits of arithmetic mean.

6. Distinguish the correlation and regression.

7. Compute Laspeyres, Paaches index numbers for the following data:

Commodity	1980		1990	
	Price Rs.	Quantity Rs.	Price Rs.	Quantity Rs.
A	5	2	10	3
B	4	5	6	6
C	6	4	10	5
D	3	4	4	6

Explain the components of time series.

SECTION B - (3 x 20 = 60 marks)

Answer any THREE questions.

All questions carry equal marks.

9. (a) In the demand function $D = \frac{1}{1+P}$, find the elasticity of demand at $P = 2$.

(b) If the revenue function is $R = -5Q^3 + 2Q^2 + 3Q$ find MR when $Q = 2$.

10. If $A = \begin{bmatrix} 2 & -3 & 0 \\ 3 & 1 & -2 \\ -1 & 0 & -4 \end{bmatrix}$ find A^{-1}.

11. Describe the uses of statistics.

12. 9 students obtained the following percentage of marks in the college test (X) and in the university exam (Y). Calculate the correlation coefficient.

X:	51	63	73	46	50	60	47	36	60
Y:	49	72	74	44	58	66	50	30	55

13. Discuss the problems in the construction of index numbers.

14. Fit a straight line trend by the method of least squares to the following data and find the trend value for the year 1985.

Year:	1978	1979	1980	1981	1982	1983	1984
Production (in 1,000 tons) :	2	5	8	10	12	15	20

OCTOBER 2005

Time: Three hours

Maximum: 100 marks

SECTION A - (4 x 10 = 40 marks)

Answer any FOUR questions.

All questions carry equal marks.

1. (a) Define a function and give an ex~ple.

   (b) Solve:

   $4x+3y=7$

   $3x+2y=9.$

2. Find $\frac{dy}{dx}$

   (a) $y = 16x^2 - 15x + 8$

   (b) $y = \frac{4x^3 - 7x^2 - 8}{4x + 3}$

3. Solve by Cramer's rule:

   $3x+2y+4z=19$

   $6x+2y+z=37$

   $x+2y+3z=10.$

4. Distinguish stratified sampling and multi stage sampling methods with suitable examples.

5. State the merits and demerits of Arithmetic mean.

6. Define regression and explain its uses.

7. Compute Laspeyre's and Paache's index numbers for the following data:

	1965		1975	
Commodity (Rs.)	Price (K.G.)	Quantity (Rs.)	Price . (K.G.)	Quantity
A	2	8	4	6
B	5	10	6	5
C	4	14	5	10
D	2	19	2	13

8. Explain the semi average method for finding trey values.

SECTION B - (3 x 20 = 60 marks)

Answer any THREE questions.

All questions carry equal marks.

9. (a) In the demand function $D = p^2 - 5p+32$, find the elasticity of demand at P = 8.

(b) If the cost functions is $C = -5Q^3 + 2Q^2 + 3Q$ find MC when Q = 2.

10. If $A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 0 & -3 \\ 3 & 4 & 0 \end{bmatrix}$ find A^{-1}.

11. Find Pearson's coefficient of Skewness from the following distribution.

Value:	6	12	18	24	30	36	42
Frequency:	4	7	9	18	15	10	5

12. (a) Define rank correlation

(b) Calculate the rank correlation coefficient for the following marks of 10 students in economics and statistics.

Economics:	80	85	75	90	88	79	70	95	65	68
Statistics:	90	75	95	85	80	70	78	89	72	83

13. Mention the tests of adequacy of good index numbers and explain any two of them.

14. Discuss about the various components of time series.

MAY 2006

1. Solve:

(a) $3x^2 + 10x + 8 = 0$

(b) $3x - 2y = 13$

$5x + 3y = 66.$

2. If the total cost function is $c = \frac{1}{3}Q^3 - 3Q^2 + 9Q$, find at what level of output AC be minimum.

3. Find the rank of $A = \begin{bmatrix} 2 & -3 & 0 \\ 2 & 6 & 8 \\ 3 & 0 & 3 \end{bmatrix}$

4. Explain the functions and limitations of statistics.

5. Find the standard deviation of the following senes :

x :	10	11	12	13	14
f:	3	12	18	12	3

6. What is a scatter diagram? How does it help in studying to correlation?

7. Explain the important differences between correlation and regression.

8. Calculate three yearly moving average of the following data:

Year:	1995	1996	1997	1998	1999	2000	2001	2002	2003	2004
No. of students :	15	18	17	20	23	25	29	33	36	40

SECTION B - (3 x 20 = 60 marks)

Answer any THREE questions.

All questions carry equal marks.

9. Explain the important properties of linearly homogenous function.

10. Find the derivatives of the following functions.:

(a) $y = (1 - \sqrt{h})(1 + \sqrt{h})$

(b) $y = \log \sqrt{1 - x^2}$

(c) $y = e^{5x^2+2x+2}$.

11. Solve the following equations by using Cramer's rule.

$$2x_1 + 3x_2 - x_3 = 9$$
$$x_1 + x_2 + x_3 = 9$$
$$3x_1 - x_2 - x_3 = -1$$

12. From the following data find out the Karl Pearson's coefficient of skewness.

x:	10	11	12	13	14	15
f:	2	4	10	8	5	1

13. Calculate rank correlation coefficient

x:	68	64	75	50	64	80	75	40	55	64
y:	62	58	68	45	81	60	68	48	50	70

14. Calculate Laspeyre's, Paasche's and Fisher's price index number.

	2003		2004	
Commodity	p_0	q_0	p_1	q_1
A	8	10	10	12
B	10	12	12	8
C	5	8	5	10
D	4	14	3	20
E	20	5	25	6

OCTOBER 2006

QUANTITATIVE TECHNIQUES

(For those who joined in July 2003 and after)

SECTION A - (4 x 10 = 40 marks)

Answer any FOUR questions.

All questions carry equal marks.

1. Define:
   (a) Variables
   (b) Functions
   (c) Equations.

2. (a) If the TC = $9Q^2 - 10Q + 50$, find the MC at Q = 20.

   (b) If the TR= $6Q^3 - Q^2 + 5Q$, find the AR and the MR.

3. Define the term 'matrix'. Explain the types of the matrices with an example.

4. Explain the limitations of statistics.

5. Calculate Quartile deviation from the following data :

Marks:	0-10	10-20	20-30	30-40	40-50	50-60
No. of students:	5	15	25	35	28	7

6. Explain the uses of Index Numbers.

7. From the following data calculate 5 yearly moving averages.

Year	1990	1991	1992	1993	1994	1995	1996	1997
Y :	78	62	68	70	66	69	72	74
Year	1998	1999	2000	2001	2002	2003	2004	
Y :	76	77	75	69	73	75	80	

8. Find Median and Mode from the following data:

C.I. :	0-10	10-20	20-30	30-40	40-50	50-60	60-70
Frequency :	6	9	10	14	9	1	1

SECTION B - (3 x 20 = 60 marks)

Answer any THREE questions.

All questions carry equal marks.

9 Solve the following simultaneous equations.

$3x+5y+z=36$

$x+2y+4z=42$

$4x+3y+2z=-8.$

10. (a) Differentiate sampling method and census method.

(b) Explain the methods of collecting secondary data.

11. Compute the two regression lines from the data given below:

X:	100	115	120	135	140	160	180	190	200	210
Y:	80	88	60	70	90	83	94	96	62	80

12. Find the Karl Pearson's coefficient of correlation between the two variables X and *Y* given below:

X:	62	72	68	58	65	70	66	63	60	72
Y:	50	65	53	59	54	48	60	64	62	71

13. From the data given below compute Paasche's and Fisher's index numbers for the year 2004 with the base year 2000.

Commodity	2000 Price	2000 Quantity	2004 Price	2004 Quantity
A	5	2	10	3
B	3	10	6	25
C	20	3	60	4
D	10	6	15	20

14. (a) Define inverse of a matrix.

(b) Find that inverse of the given matrix.

$$A = \begin{bmatrix} 5 & 1 & 1 \\ 0 & 2 & 2 \\ 3 & 1 & 4 \end{bmatrix}$$

OCTOBER 2007

SECTION A - (4 x 10 = 40 marks)

Answer any FOUR questions.

All questions carry equal marks.

1. Solve the following equations:

$$x + \frac{4}{y} = 3 \text{ and } y + \frac{4}{x} = -3.$$

2. (a) Find the equation of the straight line passes through the points.(2, 2) and (4,8).

(b) Write the equation of the straight line of gradient 2/3 and which makes negative intercept of 4 units on the y-axis.

3. Find the first order and the second order derivatives of the following functions :

(a) $Y = \dfrac{x^3 + 4x^2 + 5}{(x^2 + 1)^3}$

(b) $Y = \log(x^4 - 4x^3 + 2x^2 + 10)$.

4. Explain the importance of statistics.

5. Calculate S.D. from the following data:

C.I.	0-10	10-20	20-30	30-40	40-50
Frequency:	25	40	35	60	30

6. Find the geometric mean and the Harmonic mean from the following data:

X :	50-60	60-70	70-80	80-90	90-100	100-110
f :	5	25	45	30	8	7

7. Find the Karl Pearson's coefficient of correlation from the data givi\n below :

X:	60	65	40	55	62	66	70	72
Y:	100	120	110	130	125	140	142	150

8. (a) Explain seasonal variation.

(b) Write any three uses oftime series.

SECTION B - (3 x 20 = 60 marks)

Answer any THREE questions.

All questions carry equal marks.

9. Find the Inverse of the matrix $A = \begin{bmatrix} 2 & -3 & 0 \\ 3 & 1 & -2 \\ -1 & 0 & -4 \end{bmatrix}$

10. Fit a straight line trend to the following data by the method of least squares.

Year:	1990	1991	1992	1993	1994	1995	1996	1997
Production:	80	78	85	88	90	86	82	92

11. From the data given below find the Karl Pearson's coefficient of skewness.

Weight:	80-90	90-100	100-110	110-120	120-130
No. of persons :	5	18	35	45	28

Weight:	130-140	140-150	150-160	160-170
No. of persons :	15	8	5	3

12. Estimate X when $Y = 50$ and estimate Y when $X = 40$ for the following data.

X:	25	27	34	33	30	36	32	35	37	38
Y :	43	46	48	55	45	44	52	53	58	56

13. Calculate

(a) three yearly and

(b) 4 yearly moving averages from the following data.

Year:	1990	1991	1992	1993	1994	1995	1996	1997
Production:	50	60	55	62	70	72	75	65

14.From the data given below compute Laspeyre's and Fisher's index numbers.

Commodity	2005		2006	
	Price	Quantity	Price	Quantity
A	20	40	25	42
B	15	30	20	28
C	22	25	25	20
D	13	30	15	35
E	12	35	10	40

OCTOBER 2008

SECTION A - (4 x 10 = 40 marks)

Answer any FOUR questions.

All questions carry equal marks.

1. Solve the following pairs of simultaneous equations:

(a)
$$3x + 2y = 13$$
$$2x + 3y = 12$$

(b)
$$4x - 3y - 15 = 0$$
$$3x - 3y - 6 = 0$$

2. Find $\frac{dy}{dx}$ for the following values of y

(a) $y = (x^2 - x - 1)(x^2 + x + 1)$

(b) $y = \frac{3x}{x-5}$

3. $A = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 4 & 7 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ find AB.

4. What are the various methods of collecting primary data?

5. Find Median :

Value:	5	6	7	8	9	10	11	12	13
Frequency:	48	52	56	60	63	57	55	50	52

6. Find the Quartile deviation and its co-efficient from the following data:

Age (years) :	15	16	17	18	19	20	21
No of students:	4	6	10	15	12	9	4

7. What is index number? What are the merits and demerits of index number?

8. Explain the various components of time series.

SECTION B - (3 x 20 = 60 marks)

Answer any THREE questions.

All questions carry equal marks.

9. State and explain the important properties of linear Homogeneous functions.

10. Find the maximum and minimum values of the following function $y = \frac{1}{3}x^3 - 2x^2 + 4x + 1$.

11. If $A = \begin{bmatrix} 1 & 4 & 3 \\ 4 & 2 & 1 \\ 3 & 2 & 2 \end{bmatrix}$ find A^{-1}.

12. Distinguish between the census and sampling methods and compare their merits and demerits.

13. Calculate the mean, median and mode from the following data:

Class:	10-20	20-30	30-40	40-50	50-60	60-70	70-80	80-90
Frequency:	4	12	40	41	27	13	9	4

14. Calculate Karl Pearson's coefficient of correlation from the following data:

X:	10	20	30	40	50	60	70	80	90	100
Y :	2	4	8	10	12	14	20	28	40	50

MAY 2009

SECTION A - (4 x 10 = 40 marks)

Answer any FOUR questions.

All questions carry equal marks.

1. Explain linearly homogenous function with an example.

2. Find out $\frac{dy}{dx}$ for

(a) $y = \log(1 + x^2)$

(b) $y = \frac{x-2}{x+7}$

3. Compare between census and sampling methods of collecting data.

4. Prove that $\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} = (a-b)(b-c)(c-a)$.

5.What are the criteria for an ideal average?

6. Compare Bowley's coefficient of skewness for the following data:

	A	B
Median	19.6	24.5
First quartile	13.5	15.8
Third quartile	29.4	37.8

7. Obtain rank correlation coefficient for the following :

Price of X :	200	190	185	192	190	184	188	190
Sales ofY :	500	610	700	630	670	800	800	780

8.State the significance of time series data.

SECTION B - (3 x 20 = 60 marks)

Answer any THREE questions.

All questions carry equal marks.

9. Find maximum point and optimum output for $x = q^3 + 48 q^2 - 180 q - 800$.

10. Obtain the solution by Cramer's rule and verity it for the system

$$2x + 4y - z = 15$$
$$x - 2y + 2z = -2$$
$$3x - y + 3z = 6$$

11. Obtain medians and quartile deviation for the following data:

Wages (Rs):	100-150	150-200	200-250	250-300	300-350
Workers:	65	95	150	105	85

12. Obtain the two regression lines for the following data. Also estimate y for x = 150 and x for y = 200.

$\bar{x} = 125$ $\sigma_x = 11$ $r = 0.6$

$\bar{y} = 145$ $\sigma_x = 14$ $n = 8$

13. Obtain the trend line and estimate production in 2010.

Year:	1998	1999	2000	2001	2002	2003	2004	2005
Production (Tone) :	38	40	65	72	69	60	87	65

14. Explain the types and uses of index What are the steps in the construction of it?

MAY 2010

SECTION A - (4 x 10 = 40 marks)

Answer any FOUR questions.

All questions carry equal marks

1. (a) Solve the quadratic equation $3x^2 + 7x + 2 = 0$ by using formula.

   (b) Determine the equation of the straight line, if y-intercept is -5 and slope is $-5/8$.

2. Calculate first and second derivatives of the following functions:

   (a) $Y = x^3 - 6x^2 + 9x$

   (b) $Y = \frac{1}{\sqrt{x}}$

3. Find the value of :

   (a) $\begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

   (b) $\begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix} \times \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

4. Discuss the definition and function of statistics.

5. Calculate the Harmonic mean from the following data.

x:	6	7	8	9	10	11
f:	4	6	9	5	2	8

6. What is Skewness? What are the various methods of measuring Skewness?

7. From the following regression equations find the mean values of X and Y series.

   $8x - 10y = -66$

   $40x - 18y = 214$

8. Discuss the uses and limitations of index number of prices.

SECTION B - (3 x 20 = 60 marks)

Answer any THREE questions.

All questions carry equal marks.

9. Define homogenous functions. State and prove Euler's theorem for homogenous function.

10. Find the derivatives of the following functions.

(a) $Y = 2x^3 - x^2 + x - 2$

(b) $Y = (x^2 + 4x)(x^2 - 4x)$

(c) $Y = \frac{x-1}{x+1}$

(d) $Y = \log(x^2 + 2x)$.

11. Find the inverse of the following Matrix.

$$A = \begin{bmatrix} 5 & 1 & 1 \\ 0 & 2 & 2 \\ 4 & 7 & 6 \end{bmatrix}$$

12. Distinguish between primary data and secondary data. Discuss the various methods of collecting primary data.

13. Calculate Median and Mode for the given data:

Class :	0-10	10-20	20-30	30-40	40-50	50-60
f :	12	18	27	20	17	6

14. Calculate the coefficient of correlation from the following data:

X:	9	8	7	6	5	4	3	2
Y:	15	16	14	13	11	12	10	8